CYLINDRICAL EQUAL-AREA PROJECTION

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Abstract

Map projections have become less and less part of the expertise of cartographers in general. Perhaps that is the reason why the cartographic profession did not fare well in the war of words fought some years ago with those who supported Dr. Arno Peters. Throughout the period of that conflict and the years since, there has been very little serious mathematical discussion of this projection, in particular, when the Earth is considered as an ellipsoid instead of a sphere. The paper addresses this issue. Some of the formulae given in it are original, appearing in print for this first time in this paper. The paper also discusses the use of the projection in maps of regions of the Earth.

There are several reasons why this paper is being issued now. It could have been issued at any time in the past twenty years or more. In that period,discussion of one particular application of the equal-area projection has generated much heat but little light. That discussion centred on what has come to be called the Peters Projection because Dr. Arno Peters claims it as his own in spite of the fact that the same variant of the projection had been described long before 1974. In 1985 the cartographic world took up arms against the doctrines of Dr. Peters' New Cartography which was new only in the sense of its absurdities, for example, in the statement that Mercator's projection lacked conformality. The heaviest artillery was brought into service but there was no roar of thunder. There was the sound of a single shot. The cartographic forces had shot themselves,not in the foot but in the head. A propaganda war had been lost but the technical war had not been won. Perhaps the underlying reason is the lack of interest in map projections on the part of cartographers in general. Map projections nowadays come from computers rather than the human head.

From time to time over the past year there has been further discussion of this projection, including the question which has been asked ever since 1974 - "Is the Peters Projection truly equal-area?" Professional cartographers should never need to ask that question. When I have said that the so-called Peters

Projection can be made simply by making the equator half the length of the normal equal-area projection they look at me in total disbelief. Yet that is a fact.

Lambert conceived the cylindrical equal-area projection in 1772. Apart from the rectangular projection, it is the simplest map projection to construct. It is based on the ancient piece of geometric knowledge that the area of the surface of a sphere is equal to the area of the circumscribing cylinder. The projection of the graticule on to the cylinder forms a regular pattern of lines in which each rectangle has the same area as the corresponding quadrangle on the surface of the globe. In its overall dimensions, the height of the map will be equal to the diameter of the sphere and its breadth equal to the circumference of the sphere. The ratio of height to breadth is therefore 1:3.14 which are not very convenient proportions. Even more important are the distortions of shape. Whilst the areas correspond to those on the globe, the shapes do not. The area of the surface of the globe between 0° and 30° is equal to half the area of the hemisphere. The 30° parallel is,therefore,placed at half the distance from the equator to the globe where the 45° parallel should be. The remaining half of the projection has to accommodate 60° of latitude.

Various measures have been taken to improve the scale differences between the meridians and the parallels by allowing the cylinder to cut the sphere at a given latitude, still projecting the graticule in the same manner as before. The east/west dimension being thereby shortened, the meridians are lengthened correspondingly to compensate for this reduction. By making the standard parallels 30° , the dimensions of the map are in the ratio 1:2.36, an improvement on 1:3.14. Using 37°.07 gives a ratio of 1:2. Many people have advocated standard parallels at 45° to give a ratio of 1:1.57.

This last choice was made by Dr. Peters for the projection he claims as his own. The parallels are reduced in length by a factor of $\sqrt{2}$ and the meridians increase in length by $\sqrt{2}$. That is equivalent to using a reference sphere of radius $r\sqrt{2}$ instead of r. In that case, the standard parallels would be at 60° on the enlarged sphere and the equator would be reduced to half its length. A map of the same shape would be made if the equator in the original projection were reduced to half its length. The area of the resulting map would be half the true area. That would be rectified by enlarging each dimension by $\sqrt{2}$, which gives the same dimensions as those obtained by adopting standard parallels at 45°.

No matter which standard parallels we adopt, we are simply reducing the east-west (x) dimension and increasing the north-south (y) dimension by the same factor. As stated above, increasing the length of the y dimension is equivalent to using a sphere of greater radius. This gives a longer meridian but the spacing of the parallels remains the sines of the angles of the latitudes. We can, for this reason, readily check whether the map is truly equal-area. For example, the 30° parallel will be half-way between 0° and 90° . The space between 70° and 50° is equal to that between 0° and 10° ; that between 75° and 45° is the same as that between 0° and 15°; and that between 80° and 40° is the same as 0° to 20° and so on.

In all the above, the Earth has been treated as a sphere but in his "New Cartography" Dr. Peters tells us that his world map has to be created by calculating the area between the equator and each parallel of latitude or by extracting those areas from spheroid tables. This aspect of his work has not attracted much attention from the cartographic world. The formula for calculating these areas is given in the New Cartography as:-

$$Z = \pi b^2 \left\{ \frac{\sin\phi \sqrt{1-e^2\cos^2\phi\,(2-e^2)}}{(1-e^2)(1-e^2\cos^2\phi)} + \frac{\ln}{e} \left[\frac{e\sin\phi + \sqrt{1-e^2\cos^2\phi\,(2-e^2)}}{\sqrt{1-e^2}\,.\,\sqrt{1-e^2\cos^2\phi}} \right] \right\}$$

Dr. Peters uses the symbol ψ in place of my symbol ϕ .Why he chooses to calculate by geocentric latitude he does not explain. Geocentric latitudes have to be calculated from geodetic latitudes. In analytical geometry and in geodesy, the "reduced" or "parametric" latitude is used for calculations on the ellipse or the ellipsoid. This is the angle called the "eccentric" angle in analytical geometry. It is fundamental to the geometry of the ellipse and, therefore, the ellipsoid. For this latitude I will use the symbol u. The three latitudes are related as follows:-

$$\tan\phi = \tan B\,(1-e^2)\ ;\ \tan u = \tan B\sqrt{1-e^2}\ ;\ \tan\phi = \tan u\sqrt{1-e^2}$$

$$\tan B = \tan\phi\,(1+e'^2);\ \tan B = \tan u\sqrt{1+e'^2}\ ;\ \tan u = \tan\phi\sqrt{1+e'^2}$$

$$e^2 = \frac{a^2-b^2}{a^2}\ ;\quad e'^2 = \frac{a^2-b^2}{b^2}\ ;\qquad a = \text{semi-axis major},\quad b = \text{semi-axis minor}$$

(To use the Peters' formula, one must first calculate ϕ from B.)

$$\frac{1}{\sqrt{1-e^2\cos^2\phi}} = \frac{\sin u}{\sin\phi} \qquad \sqrt{1-e^2\cos^2\phi\,(2-e^2)} = \frac{\sin\phi}{\sin B}$$

The formula can therefore be re-written in terms of B and u.

$$Z = \pi b^2 \left\{ \frac{\sin^2 u}{(1-e^2)\sin B} + \frac{\ln}{e} \left[e.\sin u + \frac{\sin u}{(\sqrt{1-e^2})\sin B} \right] \right\}$$

Many years ago I devised the formula:-

$$Z = \pi a b \left[\sin u \sqrt{1+e'^2 \sin^2 u} + \frac{\ln}{e'} \left(e' \sin u + \sqrt{1+e'^2 \sin^2 u} \right) \right] \qquad \dagger$$

$$\left(NB,\ \pi a b = \pi b^2 \sqrt{1+e'^2} \right)$$

The formula results from the integration of:-

$$\int_0^u \cos u \sqrt{1+e'^2 \sin^2 u} \; du$$

It is also directly deduced from the formula given in the New Cartography:-

$$z = \frac{2\pi a}{b^2} \int_0^y \sqrt{b^4 + (a^2-b^2) y^2} \, . \, dy \;\Bigg\| \; \text{N.B. } \frac{1}{b^2} \sqrt{b^4 + (a^2-b^2) y^2} = \sqrt{1+e'^2 \sin^2 u}$$

After integration the formula gives:-

$$z = \frac{\pi a y}{b^2} \sqrt{b^4 + (a^2-b^2) y^2} + \frac{\pi a b}{\sqrt{a^2-b^2}} \ln \left(\frac{y\sqrt{a^2-b^2} + \sqrt{b^4+(a^2-b^2) y^2}}{b^2} \right)$$

which can be re-written as:-

$$Z = \pi a b \left[\sin u \sqrt{1+e'^2 \sin^2 u} + \frac{\ln}{e'} \left(e' \sin u + \sqrt{1+e'^2 \sin^2 u} \right) \right] \qquad \dagger$$

These same formulae may be re-written in terms of u and e as follows:-

$$Z \quad \pi b^2 \left\{ \frac{\sin u \sqrt{1-e^2 \cos^2 u}}{\sqrt{1-e^2}} + \frac{\ln}{e} \left[\frac{e \sin u + \sqrt{1-e^2 \cos^2 u}}{\sqrt{1-e^2}} \right] \right\}$$

For calculating areas in terms of B, the geodetic latitude, we have the well-known formulae:-

$$Z = \pi b^2 \left[\frac{\sin B}{1-e^2 \sin^2 B} + \frac{\ln}{e} \left(\frac{e \sin B + 1}{\sqrt{1-e^2 \sin^2 B}} \right) \right]$$

$$Z \quad \pi b^2 \left[\frac{\sin B}{1-e^2 \sin^2 B} + \frac{\ln}{2e} \left(\frac{1+e \sin B}{1-e \sin B} \right) \right]$$

$$Z \quad \pi b^2 \left[\frac{\sin B}{1-e^2 \sin^2 B} - \frac{\ln}{2e} \left(\frac{1-e \sin B}{1+e \sin B} \right) \right]$$

It does not matter which latitudes or which formulae are used, they all give the same result for the area between a given parallel and the equator. Calculation of the area gives the distance of the parallel from the equator in the projection. Although calculated on an ellipsoid, the parallels are plotted as though they were on a sphere whose surface area is the same as that of the ellipsoid. The radius of such a sphere is:- $R = a \sqrt{\frac{1}{2} \left[1 + \frac{1-e^2}{2e} \cdot \ln \left(\frac{1+e}{1-e} \right) \right]}$

When the parallels are plotted their positions are found not to be the same as latitudes B, u, or $\varnothing$ but at another latitude altogether. This latitude is known as the "authalic" latitude. If we adopt a sphere of equal-area instead of the ellipsoid, the authalic latitude should be used. Its value is very close to $B - \frac{2}{3}(B - \varnothing)$ For modern ellipsoids, the values for 45° are:-

$$B = 45° \;;\; u = 44°.9037; \quad = 44°.8076$$

Authalic latitude = 44°.8717; Radius R = 6371.118

Whether we use authalic latitudes or retain geodetic latitudes depends on the scale of the map. Taking standard parallels at 45°, the difference between the two at latitude 40° is:-

Scale 1/10M - 1.5 mm; 1/20M - 0.75 mm; 1/40M - 0.375 mm

(This represents a difference on the Earth of 15.22 km)

So far we have been considering the use of the projection for maps of the whole Earth. In those maps, distortion increases rapidly away from the equator. Varying the standard parallels does not remove the inherent distortions. Indonesia is reduced to almost half its length when the standard parallels are at 45°. Its north/south dimension is exaggerated. That is because areas are calculated as:-

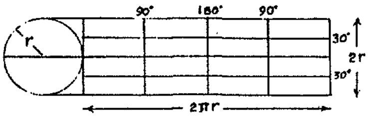
$$z = 2\pi r^2 (\sin\varnothing_1 - \sin\varnothing_2)$$

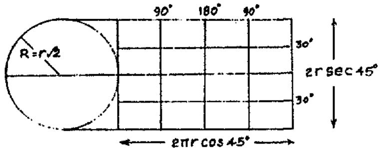
For maps of regions, as opposed to the whole Earth, we can use the formula:-

$$z = 2\pi r^2 \cos\left(\frac{\varnothing_1 - \varnothing_2}{2}\right) . 2 \sin\left(\frac{\varnothing_1 - \varnothing_2}{2}\right)$$

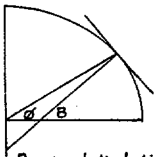
This produces a map whose x dimension is the circumference at mid-latitude and whose y dimension is the chord between the two latitudes. For maps covering 20° x 20°, the dimensions for selected latitudes are given in the Appendix. If the Earth is treated as an ellipsoid, authalic latitudes will be used.

For the polar regions, distortion is extreme and the azimuthal equal-area projection is required. Parallels are shown as concentric circles (see Appendix). Authalic latitudes can likewise be used.

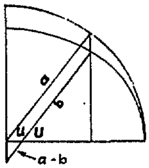




$2\pi r \cos 45° = 2\pi R \cos 60°$
$2\pi r \sec 45° = 2\pi R \sec 60°$



B = geodetic latitude
Ø = geocentric latitude

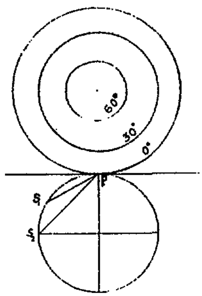


u = reduced latitude
(eccentric angle)

Regional maps - dimensions of sheets:

between 70° and 50°,	x = 1111·949 km	;	y = 2212·625 km.
60° and 40°,	x = 1429·949 km	;	y = "
40° and 20°,	x = 1925·952 km	;	y = "
20° and 0°,	x = 2190·112 km	;	y = "

Azimuthal equal-area projection



$\varnothing_1 = 90°$

$$\text{Area}(Z) = 4\pi r^2 \cos\left(\frac{\varnothing_1+\varnothing_2}{2}\right)\sin\left(\frac{\varnothing_1-\varnothing_2}{2}\right)$$

$$\cos\left(\frac{\varnothing_1+\varnothing_2}{2}\right) = \sin\left(\frac{\varnothing_1-\varnothing_2}{2}\right)$$

$$Z = 4\pi r^2 \left(\sin\frac{\varnothing_1-\varnothing_2}{2}\right)^2$$

$$R = 2r\sin\left(\frac{\varnothing_1-\varnothing_2}{2}\right) = PS$$