CONFORMAL MAP PROJECTIONS BY LEAST SQUARES ADJUSTMENT WITH CONDITIONS BETWEEN PARAMETERS

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Abstract

In this paper we apply a least squares technique to the design of conformal map projections. The mathemàtical model employed is the model with conditions between parameters, which may provide a solution to the problem of minimize the overall deformation in a region irregularly shaped.

1 Introduction

Several papers have studied the problem of adapting a well-known conformal map projection (such as stereographic projection, Mercator projection, Lambert conical conformal projection ...) to obtain an optimal conformal map projection suitable for a region with an irregular shape, [1,3,4,5]. The transformation deals with the well-known equation

$$x+iy=\sum_{j=1}^{n}\left(A_j+iB_j\right)\left(x'+iy'\right)^j, \tag{1}$$

where (x',y') are rectangular coordinates in an initial standard conformal projection, n is an integer greater than 1, A_j and B_j are real numbers, i^2 is -1 and (x,y) are rectangular coordinates in the new conformal map projection. It seems that A_j, B_j and n may be determined in such a way that the scale factor in the new projection can be adapted to any pattern in order to assure a minimum deformation in a special region. In practice [1], the problem is not so easy. In [5] it can be found a beautiful example of the determination of a new conformal map projection showing the 50 States of the United States and the passages connecting them with a scale distortion of less than ±2%. Our problem is similar to that one with a significant modification. We shall try to reduce to a minimum the deformation in a special region and to surround it by a line of constant deformation in order to fulfill the conditions of Tchebychef's principle [6]. This problem may be solved in principle, using a least squares adjustment with conditions between parameters.

2 Mathematical background

Using the deMoivre's theorem we can rewrite equation (1)

$$x+iy=\sum_{j=1}^{n}\rho^j\left[\left(A_j\cos j\theta-B_j\sin j\theta\right)+i\left(A_j\cos j\theta+B_j\sin j\theta\right)\right], \tag{2}$$

where ρ and θ are polar coordinates corresponding to (x',y'). We can separate the real and imaginary parts in equation (2) obtaining

$$x = \sum_{j=1}^{n} \rho^j \left(A_j \cos j\theta - B_j \sin j\theta \right), \tag{3}$$

$$y = \sum_{j=1}^{n} \rho^j \left(A_j \cos j\theta + B_j \sin j\theta \right). \tag{4}$$

For a conformal map projection of the sphere, the expression for the scale factor is

$$k = \frac{\sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2}}{R}, \tag{5}$$

where R is the radius of the sphere and ϕ is the latitude of the point.

Now differentiating equations (3) and (4)

$$\frac{\partial x}{\partial \phi} = \sum_{j=1}^{n} j\rho^{j-1}\left[A_j \cos(j-1)\theta - B_j \sin(j-1)\theta\right]\frac{\partial x'}{\partial \phi} - \sum_{j=1}^{n} j\rho^{j-1}\left[A_j \sin(j-1)\theta - B_j \cos(j-1)\theta\right]\frac{\partial y'}{\partial \phi}, \tag{6}$$

$$\frac{\partial y}{\partial \phi} = \sum_{j=1}^{n} j\rho^{j-1}\left[A_j \sin(j-1)\theta + B_j \cos(j-1)\theta\right]\frac{\partial x'}{\partial \phi} + \sum_{j=1}^{n} j\rho^{j-1}\left[A_j \cos(j-1)\theta - B_j \sin(j-1)\theta\right]\frac{\partial y'}{\partial \phi}. \tag{7}$$

Taking into account that the scale factor on the initial map projection is

$$k' = \frac{\sqrt{\left(\frac{\partial x'}{\partial \phi}\right)^2 + \left(\frac{\partial y'}{\partial \phi}\right)^2}}{R}, \tag{8}$$

we see that k and k' can be related (combining equations (5),(6) and (7) using complex notation)

$$k = \left| \sum_{j=1}^{n} j\left(A_j + iB_j\right)\left(x' + iy'\right)^{j-1} \right| k' . \tag{9}$$

3 Least squares adjustment with conditions between parameters

We have therefore,

$$k = \mathrm{k}(A_i, B_i, \phi, \lambda) . \tag{10}$$

Since our aim is to solve the problem using a least squares technique, we must work with a discrete distribution of points, in order to represent de continous region to map by a set of discrete points. We shall look for the coefficients A_i, B_i in order to fulfill the two requeriments

$$E = \sum_{p=1}^{m_i} \ln^2 k_p = min \qquad \text{at } m_i \text{ interior points} \tag{11}$$

$$\ln k_{m_c} = \text{const} \qquad \text{at } m_c \text{ contour points} \tag{12}$$

3.1 Interior points

In matrix notation we call

$$\mathbf{a} = \left(A_1, \ldots, A_n, B_1, \ldots, B_n\right)', \tag{13}$$

$$\mathbf{v} = \left(\ln k_1, \ln k_2, \ldots, \ln k_m\right)'. \tag{14}$$

The equations for the deformation are then

$$\mathbf{v} = \mathbf{v}(\mathbf{a}) \tag{15}$$

which can be linealized

$$\mathbf{v} = \mathbf{v}(\mathbf{a}^0) + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{a}}\right)\Bigg|_{\mathbf{a}^0} \left(\mathbf{a} - \mathbf{a}^0\right) \tag{16}$$

where $\mathbf{a}^0$ is an approximate value for the coefficients vector $\mathbf{a}$. If we call

$$\mathbf{A} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{a}}\right)\Bigg|_{\mathbf{a}^0}, \quad \mathbf{a} - \mathbf{a}^0 = \delta\mathbf{a}, \quad -\mathbf{v}(\mathbf{a}^0) = \mathbf{t}, \tag{17}$$

we can obtain a system of linear equations

$$\mathbf{A}\delta\mathbf{a} - \mathbf{t} = \mathbf{v}, \tag{18}$$

3.2 Contour points

In that case and using a notation similar to that used in *3.1*, condition (12) may be written as a condition between parameters

$$\mathbf{D}\delta\mathbf{a} - \mathbf{t}_c = 0 \tag{19}$$

3.3 Solution

System (18)-(19) has to be solved for $\delta\mathbf{a}$ under the condition $\mathbf{v}'\mathbf{v} = min$.

Approximated values $\mathbf{a}^0$ are taken in such a way that the first projection is the initial one. That is $A_1 = 1$, $A_j = 0$ if $j \neq 1$ and $B_j = 0$. Furthermore, since the effect of the coefficient B_1 is merely a rotation of the map, this coefficient will be constant B_1=0.
The elements of the two design matrices **A** and **D** are obtained from the derivatives

$$\frac{\partial v}{\partial A_q} = \frac{k'^2}{k^2} q\rho^{q-1} \sum_{j=1}^{n} j\rho^{j-1}\left[A_j \cos(q-j)\theta + B_j \sin(q-j)\theta\right], \tag{20}$$

$$\frac{\partial v}{\partial B_q} = \frac{k'^2}{k^2} q\rho^{q-1} \sum_{j=1}^{n} j\rho^{j-1}\left[-A_j \sin(q-j)\theta + B_j \cos(q-j)\theta\right]. \tag{21}$$

Now the solution to our problem is [2]

$$\delta\mathbf{a} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{t} + (\mathbf{A}'\mathbf{A})^{-1}\mathbf{D}'(\mathbf{D}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{D}')^{-1}(\mathbf{t}_c - \mathbf{D}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{t}). \qquad (22)$$

4 Examples

As an example, we have applied this technique to the design of a map of Chile using Mercator projection as the basis (n=5) and a map of the Mediterranean sea, using Lambert conformal conical projection as the basis (n=8). The resulting projections are represented in figures 1 and 3. Isocol diagrams are presented in both cases in Plate Carrée projection in figures 2 and 4.

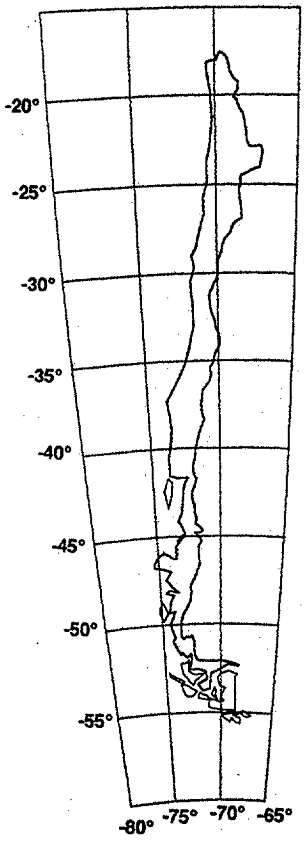

Figure 1. Chile. Complex conformal transformation of the Mercator projection with n=5 coefficients

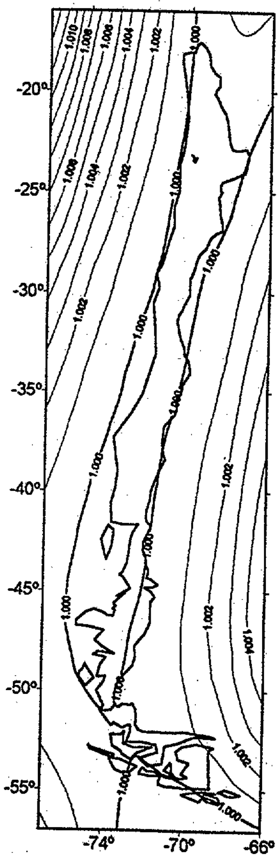

Figure 2. Chile. Plate Carrée projection to show the lines of constant scale factor for the projection of Figure 1.

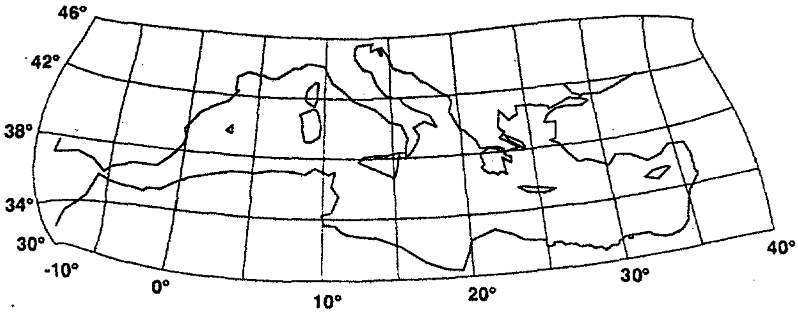

Figure 3. Mediterranean Sea. Complex conformal transformation of the Lambert conformal conical projection with n=8 coefficients.

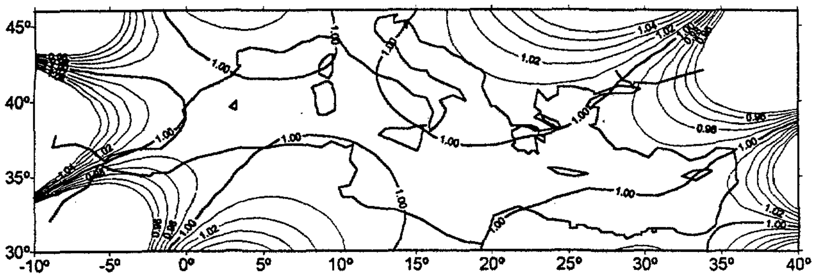

Figure 4. Mediterranean Sea. Plate Carrée projection to show the lines of constant scale factor for the projection of Figure 3.

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