CONFORMAL MAP PROJECTIONS BY LEAST SQUARES ADJUSTMENT WITH CONDITIONS BETWEEN PARAMETERS

Sergio González López
Departament d'Expressió Gràfica Arquitectònica II
Universitat Politècnica de Catalunya
Avinguda del Doctor Marañón 44-50
E-08028 Barcelona (Spain)

Abstract

In this paper we apply a least squares technique to the design of conformal map projections. The mathematical model employed is the model with conditions between parameters, which may provide a solution to the problem of minimize the overall deformation in a region irregularly shaped.

1 Introduction

Several papers have studied the problem of adapting a well-known conformal map projection (such as stereographic projection, Mercator projection, Lambert conical conformal projection ...) to obtain an optimal conformal map projection suitable for a region with an irregular shape, [1,3,4,5]. The transformation deals with the well-known equation

\[ x + iy = \sum_{j=1}^{n} (A_j + iB_j)(x' + iy')^j, \]

where \((x', y')\) are rectangular coordinates in an initial standard conformal projection, \(n\) is an integer greater than 1, \(A_j\) and \(B_j\) are real numbers, \(i^2 = -1\) and \((x, y)\) are rectangular coordinates in the new conformal map projection. It seems that \(A_j, B_j\) and \(n\) may be determined in such a way that the scale factor in the new projection can be adapted to any pattern in order to assure a minimum deformation in a special region. In practice [1], the problem is not so easy. In [5] it can be found a beautiful example of the determination of a new conformal map projection showing the 50 States of the United States and the passages connecting them with a scale distortion of less than ±2%. Our problem is similar to that one with a significant modification. We shall try to reduce to a minimum the deformation in a special region and to surround it by a line of constant deformation in order to fulfill the conditions of Tchebycheff's principle [6]. This problem may be solved in principle, using a least squares adjustment with conditions between parameters.

2 Mathematical background

Using the deMoivre's theorem we can rewrite equation (1)

\[ x + iy = \sum_{j=1}^{n} \rho^j \left[ (A_j \cos j\theta - B_j \sin j\theta) + i(A_j \cos j\theta + B_j \sin j\theta) \right], \]

where \(\rho\) and \(\theta\) are polar coordinates corresponding to \((x', y')\). We can separate the real and imaginary parts in equation (2) obtaining
\[ x = \sum_{j=1}^{\beta} \rho^j \left( A_j \cos j \theta - B_j \sin j \theta \right), \]  
(3)

\[ y = \sum_{j=1}^{\beta} \rho^j \left( A_j \cos j \theta + B_j \sin j \theta \right). \]  
(4)

For a conformal map projection of the sphere, the expression for the scale factor is

\[ k = \frac{\sqrt{\left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2}}{R}, \]  
(5)

where \( R \) is the radius of the sphere and \( \phi \) is the latitude of the point.

Now differentiating equations (3) and (4)

\[ \frac{\partial x}{\partial \phi} = \sum_{j=1}^{\beta} j \rho^{j-1} \left[ A_j \cos (j-1) \theta - B_j \sin (j-1) \theta \right] \frac{\partial x'}{\partial \phi} - \sum_{j=1}^{\beta} j \rho^{j-1} \left[ A_j \sin (j-1) \theta - B_j \cos (j-1) \theta \right] \frac{\partial y'}{\partial \phi}, \]  
(6)

\[ \frac{\partial y}{\partial \phi} = \sum_{j=1}^{\beta} j \rho^{j-1} \left[ A_j \sin (j-1) \theta + B_j \cos (j-1) \theta \right] \frac{\partial x'}{\partial \phi} + \sum_{j=1}^{\beta} j \rho^{j-1} \left[ A_j \cos (j-1) \theta - B_j \sin (j-1) \theta \right] \frac{\partial y'}{\partial \phi}. \]  
(7)

Taking into account that the scale factor on the initial map projection is

\[ k' = \frac{\sqrt{\left( \frac{\partial x'}{\partial \phi} \right)^2 + \left( \frac{\partial y'}{\partial \phi} \right)^2}}{R}, \]  
(8)

we see that \( k \) and \( k' \) can be related (combining equations (5),(6) and (7) using complex notation)

\[ k = \left| \sum_{j=1}^{\beta} j \left( A_j + iB_j \right) \left( x' + iy' \right)^{j-1} \right| k'. \]  
(9)

3 Least squares adjustment with conditions between parameters

We have therefore,

\[ k = k(A_i, B_i, \phi, \lambda). \]  
(10)

Since our aim is to solve the problem using a least squares technique, we must work with a discrete distribution of points, in order to represent the continuous region to map by a set of discrete points. We shall look for the coefficients \( A_i, B_i \) in order to fulfill the two requirements

\[ E = \sum_{p=1}^{m} \ln k_p = \text{min} \quad \text{at } m_i \text{ interior points} \]  
(11)

\[ \ln k_\alpha = \text{const} \quad \text{at } m_c \text{ contour points} \]  
(12)
3.1 Interior points

In matrix notation we call
\[ a = (A_1, \ldots, A_n, B_1, \ldots, B_m)^T, \quad (13) \]
\[ v = (\ln k_1, \ln k_2, \ldots, \ln k_m)^T. \quad (14) \]

The equations for the deformation are then
\[ v = v(a) \quad (15) \]
which can be linearized
\[ v = v(a^0) + \left( \frac{\partial v}{\partial a} \right)_{a=a^0} (a - a^0) \quad (16) \]
where \( a^0 \) is an approximate value for the coefficients vector \( a \). If we call
\[ A = \left( \frac{\partial v}{\partial a} \right)_{a=a^0}, \quad a - a^0 = \delta a, \quad -v(a^0) = t, \quad (17) \]
we can obtain a system of linear equations
\[ A \delta a - t = v. \quad (18) \]

3.2 Contour points

In that case and using a notation similar to that used in 3.1, condition (12) may be written as a condition between parameters
\[ D \delta a - t = 0. \quad (19) \]

3.3 Solution

System (18)-(19) has to be solved for \( \delta a \) under the condition \( v'v = \min \).

Approximated values \( a^0 \) are taken in such a way that the first projection is the initial one. That is
\( A_1 = 1, \quad A_j = 0 \) if \( j \neq 1 \) and \( B_j = 0 \). Furthermore, since the effect of the coefficient \( B_1 \) is merely a rotation of the map, this coefficient will be constant \( B_1 = 0 \).

The elements of the two design matrices \( A \) and \( D \) are obtained from the derivatives
\[ \frac{\partial v}{\partial a} = \frac{k^2}{k^2} q^{p+1} \sum_{j=1}^s j p^{j-1} \left[ A_j \cos(q-j)\theta + B_j \sin(q-j)\theta \right], \quad (20) \]
\[ \frac{\partial v}{\partial B} = \frac{k^2}{k^2} q^{p+1} \sum_{j=1}^s j p^{j-1} \left[ -A_j \sin(q-j)\theta + B_j \cos(q-j)\theta \right]. \quad (21) \]
Now the solution to our problem is [2]

$$\delta \mathbf{a} = (\mathbf{A}' \mathbf{A})^{-1} \mathbf{A}' \mathbf{t} + (\mathbf{A}' \mathbf{A})^{-1} \mathbf{D}' (\mathbf{D}(\mathbf{A}' \mathbf{A})^{-1} \mathbf{D}')^{-1} (t_x - \mathbf{D}(\mathbf{A}' \mathbf{A})^{-1} \mathbf{A}' \mathbf{t}).$$  \hspace{1cm} (22)

4 Examples

As an example, we have applied this technique to the design of a map of Chile using Mercator projection as the basis ($n=5$) and a map of the Mediterranean sea, using Lambert conformal conical projection as the basis ($n=8$). The resulting projections are represented in figures 1 and 3. Isocoll diagrams are presented in both cases in Plate Carrée projection in figures 2 and 4.
Figure 3. Mediterranean Sea. Complex conformal transformation of the Lambert conformal conic projection with $n=8$ coefficients.

Figure 4. Mediterranean Sea. Plate Carrée projection to show the lines of constant scale factor for the projection of Figure 3.

References


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