NEW APPROACH TO THE GAUSS-KRÜGER PROJECTION OF AN ELLIPSOID ONTO A SPHERE

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Abstract
In the paper there are presented some bases of construction and properties of the Gauss-Krüger projection of an ellipsoid onto a sphere, i.e. conformal projection of an ellipsoid, where chosen ellipsoidal meridian is projected without distortion on the spherical meridian. The consequences of such approach is that the image of ellipsoidal graticule does not cover the spherical graticule, except the equator and the central meridian of an ellipsoid, which are projected on the equator and a meridian of a sphere. Because projection is conformal one, in spite of not covering graticules, perpendicularity of images of ellipsoidal meridians and parallels is preserved. There are presented some interesting problems concerning examining the image of ellipsoidal graticule, i.e. shape of meridians and parallels images, determination of the range of image of an ellipsoid on a sphere and analysis of distortion. The presented projection may be applied as a base for creation of compound projection by means of a sphere with the given properties. Such a sphere with conformal image of an ellipsoid may be applied to construction of the area stretched along meridians. Such problems occur in creation of maps of satellite scanning.

1. Analytical description of the Gauss-Krüger projection in its fundamental version

The Gauss-Krüger projection is the conformal projection of the entire, oblate ellipsoid of revolution

$$\vec{r} = \vec{r}(B,L) = \left[\frac{a\cos B\cos l}{\sqrt{1-k^2\sin^2 B}},\ \frac{a\cos B\sin l}{\sqrt{1-k^2\sin^2 B}},\ \frac{a(1-k^2)\sin B}{\sqrt{1-k^2\sin^2 B}} \right]$$

$$(B,L) \in \omega = \left\{(B,L) : B \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle, L \in \langle -\pi, \pi) \right\}$$

$$l = L - L_0, L_0 = \text{const}, \quad k^2 = \frac{a^2-b^2}{a^2} \tag{1}$$

into a plane

$$F = x_e + iy_e = \int_0^{\vartheta} M(t)dt = w(\vartheta)$$

$$z_e = q_e + il_e = \int_0^{\vartheta} \frac{M(t)}{N(t)\cos t}dt = \psi(\vartheta) \quad , \quad \vartheta = \vartheta_1 + i\vartheta_2, \tag{2}$$

for which

$$M = M(B) = \frac{a(1-k^2)}{\left(\sqrt{1-k^2\sin^2 B}\right)^3}, \quad N = N(B) = \frac{a}{\sqrt{1-k^2\sin^2 B}}. \tag{3}$$

B – geodesic, ellipsoidal latitude,
q_e- geodesic, isometric, ellipsoidal latitude, determined by the formula

$$q_e = \ln\left[\left(\frac{1-k\sin B}{1+k\sin B}\right)^{\frac{k}{2}} \tan\left(\frac{\pi}{4} + \frac{B}{2}\right)\right], \tag{4}$$

(q_e,l_e) – isometric parameters of point (B,L) of the ellipsoid surface.
Transfer from the co-ordinates (q_e,l_e) of the point (B,L) of the ellipsoid surface to the co-ordinates x_e,y_e of the projection plane, is performed basing on the distribution of a function, consisting of the real part (Balcerzak 2003)

$$\operatorname{Re} w(\psi^{-1}(z)) \tag{5}$$

and the imaginary part

$$\operatorname{Im} w(\psi^{-1}(z)). \tag{6}$$

At first, the following relation is introduced

$$q_e = \frac{1}{2}\ln\left(\frac{1+\sin B}{1-\sin B}\right) - \frac{k}{2}\ln\left(\frac{1+k\sin B}{1-k\sin B}\right) =$$

$$= \tanh^{-1}[\operatorname{sn}(u,k)] - k\tanh^{-1}[k\operatorname{sn}(u,k)] \tag{7}$$

considering that

$$\tanh^{-1}(\sin(B)) = \frac{1}{2}\ln\left(\frac{1+\sin B}{1-\sin B}\right), \tag{8}$$

where $\operatorname{sn}(u,k)$ – means (Byrd 1954) a certain function of the argument u and the parameter k, called the Jacobi elliptic sine.
The following substitution is applied:

$$\sin B = \operatorname{sn}(u,k). \tag{9}$$

Then, the transfer from the real argument u to the complex argument $z=u+iv$ is performed. This results in the distribution of the function

$$q_e + il_e = \tanh^{-1}[\operatorname{sn}(u+iv)] - k\tanh^{-1}[k\operatorname{sn}(u+iv)] \tag{10}$$

into the real part

$$q_e = \tanh^{-1}[\operatorname{sn} u\operatorname{dn}'v] - k\tanh^{-1}[\operatorname{sn} u\operatorname{dn}'(K-u)\operatorname{tn}'v] \tag{11}$$

and the imaginary part

$$l_e = \tan^{-1}\left[\frac{\operatorname{tn}'v}{\operatorname{sn}(K-u)}\right] - k\tan^{-1}[k\operatorname{sn}(K-u)\operatorname{tn}'v]. \tag{12}$$

Now, the following notations will be introduced:

dn'v- a certain function, called, the "Jacobi amplitude delta", taken from the auxiliary module:

$$k' = \sqrt{1-k^2}\,, \tag{13}$$

$\mathrm{tn}'v = \dfrac{\mathrm{sn}'v}{\mathrm{cn}'v}$ Jacobi elliptic tangent, depending on the argument v and the auxiliary module k',

sn'v, cn'v - Jacobi elliptic sine and cosine, depending on the argument v and the auxiliary module k',

K- a quarter-period of Jacobi elliptic functions, depending on the module k of the ellipsoid surface,

K'- a quarter-period of Jacobi elliptic functions, depending on the auxiliary module k'.

The similar procedure is applied for distribution of the function $F=w(\vartheta)$ into the real and the imaginary parts.

The element of the meridian arc of the ellipsoid surface

$$ds_e = MdB = \frac{a(1-k^2)}{\left(\sqrt{1-k^2\sin^2 B}\right)^3}dB \tag{14}$$

is considered and transformed to the form

$$ds_e = \frac{ak'\,du}{\mathrm{dn}^2 u} = a\,\mathrm{dn}^2(K+u)du. \tag{15}$$

The following notations are introduced

$$\cos B = \mathrm{cn}\,u, \quad \mathrm{sn}(K'-u) = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}, \quad dB = (\mathrm{dn}\,u)du. \tag{16}$$

Integrating of both sides results in the expression

$$s_e = a\int_0^u \mathrm{dn}^2(K+u)d(K+u) = aE(K+u) - aE, \tag{17}$$

in which $E(K+u)$ i $E(k)$ mean (Byrd 1954), the, so-called, Jacobi complete eliptical integrals of the second type, taken from their arguments $K+u$ and K.

Widening the argument u with the imaginary part, i.e. assuming the complex variable $(u+iv)$ instead of u, one obtains

$$F = x_e + iy_e = a[E(K+u+iv) - E(K)]. \tag{18}$$

Distribution into the real part x_e and the imaginary part y_e leads to the distribution

$$\frac{x_e}{a} = E(u) - \frac{k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{dn}^2 u + (\mathrm{dn}'v)^2 - 1},$$

$$\frac{y_e}{a} = v - E'(v) + \frac{(k')^2\,\mathrm{sn}'v\,\mathrm{cn}'v\,\mathrm{dn}'v}{\mathrm{dn}^2 u + (\mathrm{dn}'v)^2 - 1}, \tag{19}$$

in which $E'(v)$ means the function $E(v)$ taken from the auxiliary module k'.

Variables x_e and y_e mean the coordinates in the Gauss-Krüger projection in the flat Ortho-Cartesian coordinate system.

2. Analytical description of the Gauss-Krüger projection of an ellipsoid onto a sphere

The basis for determination of the Gauss-Krüger projection of an ellipsoid onto a sphere is a function

$$q_e = f(q_k) \tag{20}$$

which relates the isometric, ellipsoidal geodesic latitude q_e and the isometric, spherical latitude q_k of the sphere surface. On the axis meridian of the ellipsoid l_e=0 that function is generated by the equality

$$s_e(B) = s_k(\varphi). \tag{21}$$

In this equality:

$s_e(B)$ – means the length of the meridian arc of the ellipsoid surface, calculated from the Equator B=0,

$s_k(\varphi)$ - the length of the axis meridian pf a sphere of the radius R, calculated from the Equator φ=0.

The value of the radius R is determined by the equality

$$R\frac{\pi}{2} = s_e\left(\frac{\pi}{2}\right). \tag{22}$$

But φ depends on the isometric spherical latitude q_k and the parameter B depends on the geodesic isometric ellipsoidal latitude q_e. Thus, the system of the following relations must occur on the axis meridian

$$w(\psi^{-1}(q_k)) = s_k, \quad w(\psi^{-1}(q_e)) = s_e \tag{23}$$

which meet the equality

$$w(\psi^{-1}(q_k)) = w(\psi^{-1}(q_e)). \tag{24}$$

After substituting the isometric latitude q_e by the complex parameter $z_e=q_e+il_e$ the right side leads to the Gauss-Krüger projection. And the left side of this equality, after substituting the parameter q_k with the complex parameter $z_k=q_k+il_k$ also describes a certain, conformal projection. In fact, it is the transverse Mercator projection. In this projection, the Equatorial semi-axis a is substituted by the sphere parameter R, with the simultaneous zeroing the eccentricity value k. Therefore, reversing the first function, with the assumption that $s_k=s_e$ allows to find the argument q_k from a given value s_e.

After widening the parameter q_k with the imaginary part l_k, after reversing s_k , one may find

$$z_k = q_k + il_k = \ln\tan\left(\frac{\pi}{4} + \frac{F}{2R}\right). \tag{25}$$

The variable F means the point (x_e,y_e), i.e. the point in the plane of the complex variable. $z_e=x_e,+y_e$.

Determination of coordinates q_k , l_k which occur in z_k consists of the distribution of the function into the real and imaginary parts.

In order to reach this, the following relation is derived:

$$\left[e^{z_k} = \tan\left(\frac{\pi}{4} + \frac{F}{2R}\right) = \frac{\sin\left(\frac{\frac{\pi}{2} + \frac{F}{R}}{2}\right)}{\cos\left(\frac{\frac{\pi}{2} + \frac{F}{R}}{2}\right)} = \sqrt{\frac{1 - \cos\left(\frac{\pi}{2} + \frac{F}{R}\right)}{1 + \cos\left(\frac{\pi}{2} + \frac{F}{R}\right)}} = \right.$$

$$\left. = \sqrt{\frac{1 + \sin\left(\frac{F}{R}\right)}{1 - \sin\left(\frac{F}{R}\right)}} \right] \Leftrightarrow \left[(e^{z_k})^2 = \frac{1 + \sin\left(\frac{F}{R}\right)}{1 - \sin\left(\frac{F}{R}\right)} \right], \tag{26}$$

and then

$$\left[(e^{z_k})^2 - (e^{z_k})^2 \sin\left(\frac{F}{R}\right) = 1 + \sin\left(\frac{F}{R}\right) \right] \equiv \left[\sin\left(\frac{F}{R}\right) = \frac{e^{2z_k} - 1}{e^{2z_k} + 1} = \tanh z_k \right]. \tag{27}$$

Then (König 1951) the following identity is used

$$\arcsin w = \frac{1}{i} \ln\left(iw + \sqrt{1 + w^2} \right). \tag{28}$$

Assuming w=tanhz_{kl} one may find

$$\left\{ \frac{F}{R} = \frac{1}{i} \ln\left[i \tanh z_k + \sqrt{1 - (\tanh z_k)^2} \right] = \frac{1}{i} \ln\left[\frac{i \sinh z_k + 1}{\cosh z_k} \right] = \right.$$

$$= \frac{1}{i} \ln\left[\frac{i \sin(iz_k) + 1}{\cos(iz_k)} \right] = \frac{1}{i} \ln\left[\frac{1 + \sin(iz_k)}{\sqrt{1 - \sin^2(iz_k)}} \right] =$$

$$\left. = \frac{1}{i} \ln \sqrt{\frac{1 + \sin(iz_k)}{1 - \sin(iz_k)}} = \frac{1}{i} \ln \tan\left(\frac{\pi}{4} + \frac{iz_k}{2} \right) \right\} \equiv$$

$$\equiv \left\{ \frac{F}{R} = \frac{1}{i} \ln \tan\left(\frac{\pi}{4} + \frac{iz_k}{2} \right) \right\} \tag{29}$$

This relation may have various forms. If, for example, it is assumed that

$$\xi = \tan\left(\frac{\pi}{4} + \frac{iz_k}{2} \right) = \sqrt{\frac{1 + \sin(iz_k)}{1 - \sin(iz_k)}} = e^{i\frac{F}{R}} \tag{30}$$

then

$$\tanh\left(\frac{i\frac{F}{R}}{2} \right) = \frac{\xi - 1}{\xi + 1} = \frac{e^{iz_k} - 1}{e^{iz_k} + 1} = i \tanh\left(\frac{z_k}{2} \right) = \tan\left(\frac{iz_k}{2} \right) \tag{31}$$

This means, that the following relation occurs

$$\tanh\left(\frac{z_k}{2} \right) = \tan\left(\frac{F}{2R} \right). \tag{32}$$

But also, another relation occurs

$$\sinh z_k = \tan\left(\frac{F}{R}\right), \tag{33}$$

since

$$\sin(iz_k) = i\sinh z_k = \frac{2\tan\left(\frac{iz_k}{2}\right)}{1+\tanh^2\left(\frac{iz_k}{2}\right)} = \tanh\left(i\frac{F}{R}\right) = i\tan\left(\frac{F}{R}\right), \tag{34}$$

and also

$$\frac{1}{\cosh z_k} = \cos\left(\frac{F}{R}\right), \tag{35}$$

since

$$\frac{1}{\cos(iz_k)} = \frac{1}{\cosh z_k} = \frac{1+\tan^2\left(\frac{iz_k}{2}\right)}{1-\tan^2\left(\frac{iz_k}{2}\right)} = \frac{1+\tanh^2\left(\frac{i\frac{F}{R}}{2}\right)}{1-\tanh^2\left(\frac{i\frac{F}{R}}{2}\right)} = \cosh\left(i\frac{F}{R}\right) = \cos\left(\frac{F}{R}\right). \tag{36}$$

The relation

$$\tanh z_k = \sin\left(\frac{F}{R}\right), \tag{37}$$

results from the fact, that

$$\tan(iz_k) = i\tanh z_k = i\sin\left(\frac{F}{R}\right) = \sinh\left(i\frac{F}{R}\right). \tag{38}$$

Determination of the argument $z_k=q_k+il_k$, expressed by variables $x_k=x_e$, $y_k=y_e$, requires that the following identity is considered (König 1951)

$$\ln\tan\hat{z} = -\operatorname{arctanh}\left(\frac{\cos 2\hat{x}}{\cosh 2\hat{y}}\right) + i\arctan\left(\frac{\sinh 2\hat{y}}{\sin 2\hat{x}}\right), \tag{39}$$

taken for

$$\left(\hat{z} = \hat{x}+i\hat{y} = \frac{\pi}{4} + \frac{F}{2R} = \frac{\pi}{4} + \frac{x_e}{2R} + i\frac{y_e}{2R}\right) \Rightarrow \left(2\hat{x} = \frac{\pi}{2} + \frac{x_e}{R}, 2\hat{y} = \frac{y_e}{R}\right). \tag{40}$$

Then one can find

$$\hat{z} = q_k + il_k = \operatorname{arctanh}\left(\frac{\sin\left(\frac{x_e}{R}\right)}{\cosh\left(\frac{y_e}{R}\right)}\right) + i\arctan\left(\frac{\sinh\left(\frac{y_e}{R}\right)}{\cos\left(\frac{x_e}{R}\right)}\right) \Leftrightarrow$$

$$\Leftrightarrow \left[\tanh q_k = \frac{\sin\left(\frac{x_e}{R}\right)}{\cosh\left(\frac{y_e}{R}\right)}, \quad \tan l_k = \frac{\sinh\left(\frac{y_e}{R}\right)}{\cos\left(\frac{x_e}{R}\right)}\right]. \tag{41}$$

If it is assumed, that

$$\left[\hat{z} = \frac{\pi}{4} + i\frac{z_k}{2} = \left(\frac{\pi}{4} - \frac{l_k}{2}\right) + i\frac{q_k}{2}\right] \Rightarrow \left[2\hat{x} = \frac{\pi}{2} - l_k,\ 2\hat{y} = q_k\right], \tag{42}$$

then

$$\frac{F}{R} = \frac{x_e}{R} + i\frac{y_e}{R} = \arctan\left(\frac{\sinh q_k}{\cos l_k}\right) + i\,\mathrm{arctanh}\left(\frac{\sin l_k}{\cosh q_k}\right). \tag{43}$$

So, the following relation occurs

$$\tan\left(\frac{x_e}{R}\right) = \frac{\sinh q_k}{\cos l_k} = \frac{\tan\varphi}{\cos l_k},\ \tanh\left(\frac{y_e}{R}\right) = \frac{\sin l_k}{\cosh q_k} = \sin l_k \cos\varphi. \tag{44}$$

Therefore, on the axis meridian

$$\tanh\left(\frac{q_k}{2}\right) = \tan\left(\frac{\varphi}{2}\right), \quad \tanh q_k = \sin\varphi, \tag{45}$$

and also

$$\sinh q_k = \tan\varphi, \quad \frac{1}{\cosh q_k} = \cos\varphi. \tag{46}$$

The above relations determine the Transverse Mercator Projection.

3. The cartographic graticule in the Gauss-Krüger projection of an ellipsoid onto a sphere

Basing on the presented method of projection of an ellipsoid onto a sphere, investigations concerning the location of images of parametric lines B=const, L=const o a sphere.

In order to facilitate visualisation of obtained results, the sphere was additionally projected onto the plane by means of the following, simple formulae $x=\varphi$, $y=\lambda$. Figure 1 presents the image of the graticule of parametric lines B=const, L=const of the entire ellipsoid, in the assumed coordinate system $x=\varphi$, $y=\lambda$. . Images of ellipsoidal meridians and parallels B=const , L=const, does not cover the geographic graticule φ=const, λ=const; they create certain curves of the location close to spherical meridians and parallels. Within the surroundings of points of coordinates B=0°, L=±90° relatively high differences in shapes of parametric lines B=const , L=const and φ=const, λ=const occur. Basing on analysis of obtained results, it may be noticed that the surroundings of points B=0°, L=±90° are some particularities of the projection.

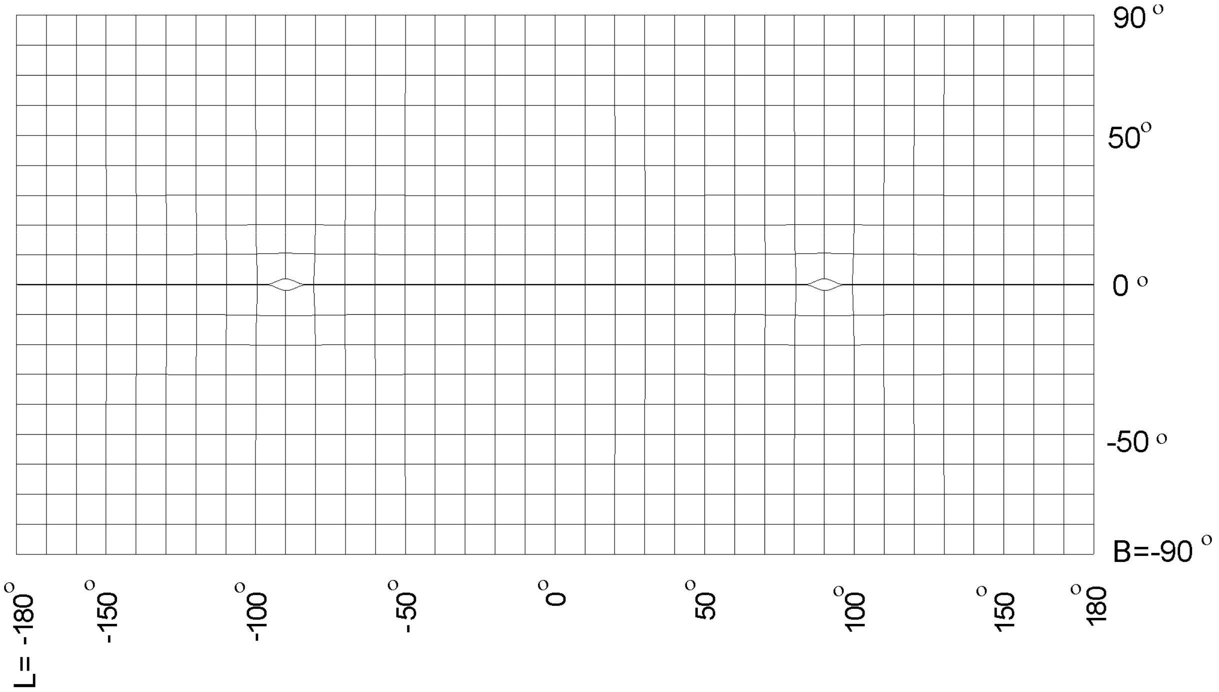

Fig. 1 The image of the graticule of parametric lines *B*=const , *L*=const in the Gauss-Krüger projection of the entire ellipsoid onto the sphere, in the φ, λ coordinate system

In order to perform detailed analysis of location of parametric lines *B*=const, *L*=const within the surrounding of the projection peculiar points, the cartographic graticule has been developed, which covers the sub-area limited by meridians between *L*=70° and *L*=90° and the parallels between *B*=0° to *B*=5°. Fig.2 presents location of this graticule on a plane, assuming for the needs of presentation and similarly to the previous case, the following method of projection a sphere onto a plane $x=\varphi, y=\lambda$.

The parametric lines φ=const, λ=const are marked as light grey and the parameteric lines *B*=const , *L*=const are marked in black. The thickened black line marks the location of the characteristic meridian of the geodesic longitude $L = 90°\left(1-k^2\right) = 82.64°$.

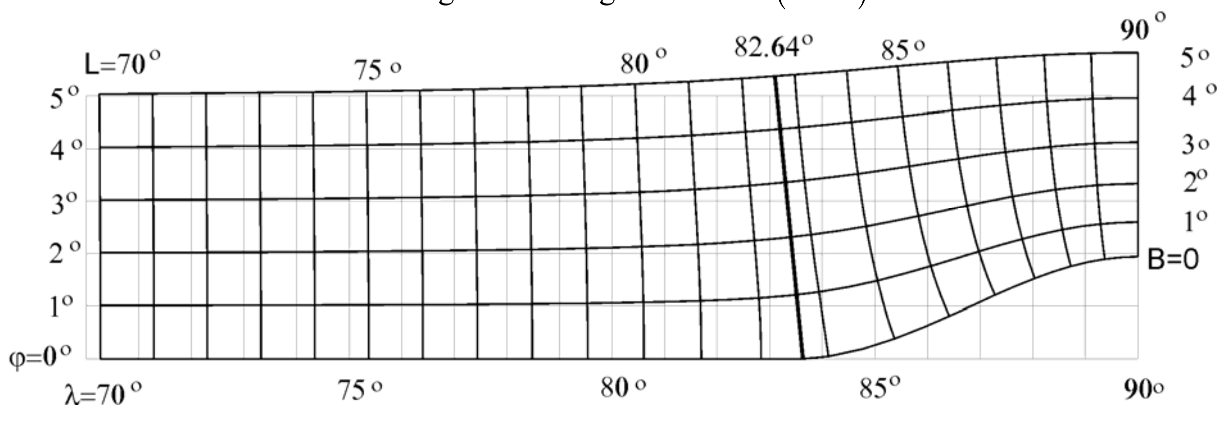

Fig.2 The image of the parametric lines *B*=const , *L*=const on the background of the graticule φ=const, λ=const, covering a fragment of an ellipsoid, limited by meridians *L*=70° and *L*=90° and parallels *B*=0° and *B*=5°.

It turns out from analysis and the Figure that meridians and parallels of the ellipsoidal coordinate system *B*=const, *L*=const, within the surroundings of points of coordinates *B*=0°, *L*=±90°, are projected on a sphere on certain curves, which do not overlap parallels and meridians $\varphi = \text{const}$, $\lambda = \text{const}$ of the sphere. The section of the ellipsoidal parallel B=0° is projected in irregular way, since at the section $L = 90° \mp (1-k^2)$ of the geodesic longitude it is projected onto two parts, which are the projection peculiarity.

4. Distribution of projection deformations in Gauss-Krüger projection of an ellipsoid onto a sphere

In the cartographic, conformal projection of an ellipsoid onto a sphere surface

$$\vec{\tau} = \vec{\tau}(\varphi, l_k) = [R\cos\varphi\cos l_k, R\cos\varphi\sin l_k, R\sin\varphi],$$

$$l_k = \lambda - \lambda_0 \; \lambda_0 = \text{const},$$

$$(\varphi,\lambda) \in \omega_k = \left\{(\varphi,\lambda) : \varphi \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2}\right\rangle, \lambda \in \langle -\pi, \pi)\right\}, \tag{47}$$

Of the radius R, which meets the condition, where

$$q_e = \int_0^B \frac{M(t)}{N(t)\cos t}\,dt = \psi_e(B) \tag{48}$$

means the geodesic, isometric ellipsoidal latitude of the ellipsoid surface, and

$$q_k = \int_0^\varphi \frac{dt}{\cos t} = \psi(\varphi), \tag{49}$$

means the isometric, spherical latitude.

Therefore, the formula, which describes the local scale of length

$$\vec{\mu} = \frac{d\vec{\tau}}{|d\vec{r}|} \tag{50}$$

in the plane of the complex variable z_e, may be expressed by means of the relation

$$\mu = |\vec{\mu}| = \frac{R\cos\varphi}{N\cos B}\left|\frac{dz_k}{dz_e}\right| = \frac{R\cos\varphi}{N\cos B}\left|\frac{dz_k}{dF}\,\frac{dF}{dz_e}\right|. \tag{51}$$

The right side may be presented in the form of the quotient

$$\mu = \frac{1}{\dfrac{1}{R\cos\varphi}\left|\dfrac{dF}{dz_k}\right|}\,\frac{1}{N\cos B}\left|\frac{dF}{dz_e}\right| = \frac{\mu_{GK}}{\mu_{PM}}, \tag{52}$$

where

$$\mu_{GK} = \frac{1}{N\cos B}\left|\frac{dF}{dz_e}\right| \tag{53}$$

means the local scale of length in the Gauss-Krüger projection, and

$$\mu_{PM} = \frac{1}{\cos\varphi}\left|\frac{dF}{dz_k}\right|, \tag{54}$$

means the local scale of length in the transverse Mercator projection.

The variable F, which occurs here, determines the complex length of a meridian arc, $s_e=s_k$, common for the surface of an ellipsoid and for the surface of a sphere.

The derivative $\frac{dF}{dz_e}$ is calculated from the relation

$$\frac{dF}{dz_e} = \frac{dF}{dt}\frac{dt}{dz_e} = M(\hat{B})\frac{N(\hat{B})\cos\hat{B}}{M(\hat{B})} = N(\hat{B})\cos(\hat{B}) \text{ dla } \hat{B} = B_1 + iB_2. \tag{55}$$

And the derivative $\frac{dF}{dz_k}$ is calculated by analogy to the above.

As it turns out from numerical experiments, for limited projection zones, the Gauss-Krüger projection of the ellipsoid surface and the transverse Mercator projection of the corresponding zone, are mutually equivalent. Differences of values of linear deformations on a meridian located within the distance of 5º from the central meridian (10-degree projection zone), do not exceed 10 mm/km in medium geographic latitude. They are very small for narrow projection zones (3º and 6º). The smallest differences of deformations occur in latitudes close to 30º. The inversion of a sign may be observed in that cases. For the meridian, which is located within the distance of 10º from the central meridian, those differences reach 50 to 40 mm/km. The biggest differences occur within the circum-equatorial areas.

Table 1. Distribution of deformations on the Gauss-Krüger projection of an ellipsoid onto a sphere

B°	l_e°	φ°	l_k°	μ_{GK}	μ_{PM}	$(\mu_{GK} - \mu_{PM})$ mm / km
0.0000	5.0000	0.0000	5.0084	1.0038457	1.0038327	13.0
0.0000	10.0000	0.0000	10.0170	1.0155330	1.0154797	53.3
10.0000	5.0000	9.9509	5.0082	1.0037283	1.0037173	11.0
10.0000	10.0000	9.9512	10.0164	1.0150507	1.0150053	45.4
20.0000	5.0000	19.9076	5.0074	1.0033909	1.0033848	6.1
20.0000	10.0000	19.9082	10.0149	1.0136670	1.0136423	24.7
30.0000	5.0000	29.8754	5.0063	1.0028752	1.0028752	0.0
30.0000	10.0000	29.8761	10.0126	1.0115614	1.0115617	0.3
40.0000	5.0000	39.8581	5.0049	1.0022451	1.0022500	−4.9
40.0000	10.0000	39.8588	10.0098	1.0090017	1.0090220	−20.3
50.0000	5.0000	49.8580	5.0035	1.0015773	1.0015845	−7.2
50.0000	10.0000	49.8584	10.0069	1.0063052	1.0063343	−29.1
60.0000	5.0000	59.8750	5.0021	1.0009525	1.0009589	−6.4
60.0000	10.0000	59.8752	10.0042	1.0037968	1.0038226	−25.8
70.0000	5.0000	69.9071	5.0010	1.0004449	1.0004487	−3.8
70.0000	10.0000	69.9072	10.0019	1.0017697	1.0017848	−15.1
80.0000	5.0000	79.9505	5.0003	1.0001146	1.0001157	−1.1
80.0000	10.0000	79.9506	10.0005	1.0004550	1.0004594	−4.4
85.0000	5.0000	84.9749	5.0001	1.0000289	1.0000291	−0.2
89.0000	5.0000	88.9950	5.0000	1.0000012	1.0000012	0.0

5. Final Remarks

The paper presents the method of the Gauss-Krüger projection of an ellipsoid onto a sphere, i.e. such conformal projection of an ellipsoid onto a sphere, for which the selected central meridian is projected without linear deformations. It is the complex projection, which consists of several partial projections. The most important stage is the conceptual Gauss-Krüger projection of an auxiliary surface, and then the use of the transverse Mercator projection of the corresponding area of a plane onto a sphere of the appropriately selected radius. The developed method allows for conformal projection of the entire ellipsoid, with maintenance of equi-distance of the central meridian. Performed numerical analyses proved that meridians L =const and parallels B=const of the ellipsoid do not overlap with meridians λ=const and parallels φ=const of the sphere. Values of deviations depend on the value of the ellipsoid flattening. For the Earth ellipsoid those values are not big, but it should be realised that they occur. Bigger deviations occur around peculiar points of the Gauss-Krüger projection. For those places the curvature of meridians, as well as parallels, is clearly visible. Although the proposed method has theoretical value, it may be practically applied in some cases.

References

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