APPLICATION OF THE CHEBYSHEV POLYNOMIALS TO APPROXIMATION AND CONSTRUCTION OF MAP PROJECTIONS

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Abstract
Usually to approximation of map projection the least square method is used. Determination of polynomials coefficients requires solution of complicated system of equations. It is possible to avoid such problem using orthogonal Chebyshev polynomials. It is completely different method of approximation, where the maximum difference between value of function and value calculated from polynomial is minimized. In the paper there are presented properties of Chebyshev polynomials, their application to map projection approximation and comparison with other methods of map projection approximations. Moreover Chebyshev polynomials may be used as a method of minimization of map projection distortion. The example of such projection is shown in the paper.

1. Uniform approximation

Approximation performed with the use of Chebyshev polynomials is called „the uniform approximation". It consists of approximation of the function $f(x)$ by the polynomial $T_n(x)$ in the interval $x \in \langle a,b \rangle$, in such a way, that the deviation of the highest absolute value

$$E = \max_{x \in \langle a,b \rangle} \left| f(x) - T_n(x) \right| \tag{1}$$

would have the possibly smallest value for the appropriate selection of coefficients of the $T_n(x)$ polynomial.

2. Chebyshev polynomials

Chebyshev polynomials of the first type and the n order may be presented in the following form (Paszkowski 1975)

$T_0 = 1$

$T_1 = x$

$T_2 = 2x^2 - 1$

$$T_j = 2xT_{n-1} - T_{n-2} \tag{2}$$

Chebyshev polynomials of the second type have the following form

$U_0 = 1$

$U_1 = 2x,$

$U_2 = 4x^2 - 1,$

$$U_n = 2xU_{n-1} - U_{n-2} \tag{3}$$

2.1 Properties of Chebyshev polynomials

The $T_n(x)$ polynomials has n zeros in the interval $\langle -1,1 \rangle$

$$x_k = \cos\left(\frac{2k+1}{n} \frac{\pi}{2} \right) \quad \text{k=0,1,...n-1} \tag{4}$$

and n+1 extremes in this interval

$$x'_k = \cos\frac{k\pi}{n}, \text{ k=0,1,...n} \quad (5)$$

Polynomials are orthogonal in the continuous case, with the scalar product

$$\left(T_i, T_j\right) = \int_{-1}^{1} T_i(x) T_j(x) \frac{1}{\sqrt{1-x^2}} dx \quad (6)$$

where

$$\left(T_i, T_j\right) = \begin{cases} 0 \quad dla \ i \neq j \\ \frac{\pi}{2} \quad dla \ \ i = j \neq 0 \\ \pi \quad dla \ \ i = j = 0 \end{cases} \quad (7)$$

And, in the discrete case, with the scalar product

$$\left(T_i, T_j\right) = \sum_{k=0}^{m} T_i(x_k) T_j(x_k), \quad x_k = \cos\left(\frac{2k+1}{m+1}\frac{\pi}{2}\right) \quad (8)$$

where x_k are zeros of the $T_{m+1}(x)$ polynomial, and

$$\left(T_i, T_j\right) = \begin{cases} 0 \quad dla \ i \neq j \\ \frac{1}{2}(m+1) \quad dla \ \ i = j \neq 0 \\ m+1 \quad dla \ \ i = j = 0 \end{cases} \quad (9)$$

If the *t* variable is introduced, which falls in the interval $\langle a, b \rangle$, then the following substitution is applied

$$t = 0.5(a+b) + 0.5(b-a)x \quad (10)$$

The derivative of Chebyshev polynomial T_n is calculated basing on the relation

$$T_n' = nU_{n-1} \quad (11)$$

According to the, so-called, minimax rule, the polynomial $2^{1-n}T_n$ has the smallest maximum norm in the interval $\langle -1, 1 \rangle$ out of all polynomials of the *n* order with the leading coefficient 1.

3. Chebyshev series of one and two variables

The function *f*(*x*) may be approximated by Chebyshev series in the form (Bjorck, Dahlquist 1987)

$$f(x)_m \approx \frac{1}{2}c_0 + \sum_{j=1}^{m} c_j T_j \quad (12)$$

In the discrete case coefficient of the series are calculated from the formulae:

$$c_j = \frac{2}{m+1}\sum_{k=0}^{m} f(x_k) T_j(x_k) \quad (13)$$

The *F*(x,y) function may be approximated by means of Chebyshev series in the form (Leng 1997)

$$F(x,y) \approx \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} T_i(x) T_j(y) \quad (14)$$

Coefficient of the series (14) are calculated using the formula

$$c_{ij} = \frac{\varepsilon}{(n+1)(m+1)}\sum_{k=0}^{n}\sum_{l=0}^{m} F(x_k, y_l)\cos\left(\frac{i(2k+1)\pi}{2(n+1)}\right)\cos\left(\frac{j(2l+1)\pi}{2(n+1)}\right) \quad (15)$$

where ε=4 for $i \neq 0$ and $j \neq 0$, ε=2 for i=0 and $j \neq 0$ or $i \neq 0$ and j=0 and ε=1 for i=0 and j=0.

Formulae (12)-(15) may be used in the case of unevenly distributed junction points. In practice it often happens that within the interval under consideration, points are evenly distributed and that calculated values of the function exist in those points. Then, the approach presented by Leng (1997) may be applied. In the case of one variable, if the node x_k is located between points x_a,x_b, for which the function values f_a, f_b were calculated, then the interpolated value may be calculated using the formula

$$f(x_k) = wf_a + (1-w)f_b \tag{16}$$

where

$$w = 0.5 - 1/[(x_b - x_a)(x_k - 0.5(x_a - x_b))] \tag{17}$$

And, in the case of two variables, the value of the function in the junction point is calculated in the following way: if the junction point (x_k,y_k) falls within the rectangle, delineated by points x_a,x_b and y_c,y_d, for which values f_a, f_b, f_c, f_d were measured, then the following formula may be used for calculation the value of the function in the junction point

$$\begin{aligned} f(x_k, y_k) = {} & 0.25(f_a + f_b + f_c + f_d) + \\ & + 0.5(f_d - f_c + f_b - f_a)/(x_b - x_a)(x_k - 0.5(x_a + x_b)) + \\ & + 0.5(f_d - f_b + f_c - f_a)/(y_b - y_a)(x_k - 0.5(y_a + y_b)) \end{aligned} \tag{18}$$

4. Utilisation of Chebyshev polynomials for approximation of cartographic projections

Utilisation of Chebyshev polynomials for approximation of cartographic projections will be presented using the simple example of the azimuthal, conformal projection of the form

$$x = R\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\cos\lambda \tag{19}$$

$$y = R\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\sin\lambda$$

where the parameters φ,λ define the geographic co-ordinates on a sphere.

The scale of lengths for this projection is presented by the formula

$$m = \frac{\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\cos\varphi} \tag{20}$$

The function (20) was approximated by the series of the form

$$m_p \approx \frac{1}{2}c_0 + \sum_{j=1}^{n} c_j T_j \tag{21}$$

in the interval $\varphi \in \langle 48°, 54° \rangle$ which covers the area of Poland.

Coefficients of the series (21) were determined in such a way that the absolute, maximum value of the error, smaller than $1*10^{-9}$ could be obtained. This condition is met for the series of the 6 order, of the coefficients

c_0	0.66942056584
c_1	0.03870692874
c_2	0.00198673489
c_3	0.00009564569
c_4	0.00000442051
c_5	0.00000019863
c_6	0.00000000873

The scale values, calculated for points, located within equal distances, by means of the formula (20) and the series (21) are listed in Table 1

Table 1

φ	m	m_p	$m-m_p$
48°	3.89324451749	3.89324451714	3.5E-10
49°	4.07680006789	4.07680006756	3.3E-10
50°	4.27431608521	4.27431608554	-3,3E-10
51°	4.48724199273	4.48724199235	3,8E-10
52°	4.71722041101	4.71722041135	-3,4E-10
53°	4.96611890326	4.96611890290	3,6E-10
54°	5.23606797750	5.23606797789	-3,9E-10

The projection (19) was approximated by means of the series

$$x \approx \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} T_i(\varphi) T_j(\lambda) \quad (22)$$

$$y \approx \sum_{i=0}^{n}\sum_{j=0}^{m} c'_{ij} T_i(\varphi) T_j(\lambda) \quad (23)$$

Assuming the accuracy of the order of 0.1 mm the following coefficients c_{ij} of the 5th order polynomial were obtained for the co-ordinate *x*

	i=0	1	2	3	4	5
j=0	17171.8435979	-520.331358448	-39.6190969591	0.19989368391	0.00760811149	-0.000023030137
1	1430.42159388	-43.3438149406	-3.30028697834	0.01665122561	0.00063375880	-0.00000191842
2	52.9630401760	-1.60485567465	-0.12219700302	0.00061653119	0.00002346566	-0.00000007103
3	2.04262039083	-0.06189431185	-0.00471275987	0.00002377770	0.00000090499	-0.00000000274
4	0.07865136626	-0.00238324860	-0.00018146544	0.00000091556	0.00000003485	-0.00000000011
5	0.00302511847	-0.00009166540	-0.00000697959	0.00000003521	0.00000000134	-0.000000000004

Table 2 presents the co-ordinate values, calculated using the formula (20) and the series (21) and the differences between those values

Table 2

φ[°]	λ[°]	x[m]	x_p[m]	$x-x_p$[m]
48	14	16104.0199	16104.0198	0.0001
52	14	17953.1174	17953.1174	0.0000
54	14	19025.4827	19025.4826	-0.0001
48	18	15784.7063	15784.7062	0.0001
52	18	17597.1396	17597.1395	0.0001
54	18	18648.2418	18648.2417	0.0001
48	22	15388.4912	15388.4911	0.0001
52	22	17155.4302	17155.4302	0.0000
54	22	18180.1485	18180.1484	0.0001

In the case of the *y* function approximation, the following coefficients c_{ij} were obtained

	i=0	1	2	3	4	5
j=0	5414.26148924	1650.28023196	-12.4918532876	-0.63398176893	0.00239882834	0.00007304224
1	451.010196137	137.469018180	-1.04057648722	-0.05281094061	0.00019982338	0.00000608445
2	16.6991824228	5.08995191626	-0.03852856706	-0.00195538713	0.00000739870	0.00000022528
3	0.64403573537	0.19630367777	-0.00148592748	-0.00007541322	0.00000028534	0.00000000869
4	0.02479868052	0.00755869888	-0.00005721583	-0.00000290380	0.00000001099	0.00000000033
5	0.00095381619	0.00029072552	-0.00000220066	-0.00000011169	0.00000000042	0.000000000013

Table 3 presents the co-ordinate values, calculated using the formula (20) and the series (21) and differences between those values

Table 3

φ[°]	λ[°]	y[m]	y_p[m]	y-y_p[m]
48	14	4015.1831	4015.1831	0.0000
52	14	4476.2149	4476.2149	0.0000
54	14	4743.5856	4743.5856	0.0000
48	18	5128.7620	5128.7620	0.0000
52	18	5717.6572	5717.6572	0.0000
54	18	6059.1811	6059.1810	0.0001
48	22	6217.3540	6217.3540	0.0000
52	22	6931.2437	6931.2437	0.0000
54	22	7345.2568	7345.2567	0.0001

In the case of approximation of the cartographic projection using Chebyshev series (22), (23) distortions will be determined by means of calculation of partial derivatives

$$x_\varphi \approx \sum_{i=1}^{n}\sum_{j=0}^{m} i c_{ij} U_{i-1}(\varphi) T_j(\lambda) \tag{24}$$

$$y_\varphi \approx \sum_{i=1}^{n}\sum_{j=0}^{m} i c'_{ij} U_{i-1}(\varphi) T_j(\lambda)$$

$$x_\lambda \approx \sum_{i=0}^{n}\sum_{j=1}^{m} c_{ij}\, j T_i(\varphi) U_{j-1}(\lambda)$$

$$y_\lambda \approx \sum_{i=0}^{n}\sum_{j=1}^{m} c'_{ij}\, j T_i(\varphi) U_{j-1}(\lambda)$$

Derivatives (24) are the basis for calculation of scales of distortions of distances, areas and angles.

5. Comparison of results obtained in using the uniform and mean square approximation

The obtained results of the uniform approximation were compared with the results of mean square approximation. Orthogonal polynomials were applied for the mean square approximation. The function $f(x)$ was approximated by the series

$$f(x) \approx y_m = \sum_{j=0}^{m} b_j^{(m)} p_j(x), \tag{25}$$

where

$$p_j(x) = \sum_{k=0}^{j} a_{kj} x_i^k \tag{26}$$

is the exponential, orthogonal polynomial. Such polynomials nay be determined using the recurrence relation (Ralston 1971)

$$p_{j+1}(x) = (x - a_{j+1})\, p_j(x) - b_j\, p_{j-1}(x), \quad (j = 0,1,...), \tag{27}$$

where

$p_0(x) = 1, \quad p_{-1}(x) = 0,$

$$\alpha_{k+1} = \frac{\sum_{i=1}^{n} x_i p_k^2(x_i)}{\sum_{i=1}^{n} p_k^2(x_i)} \tag{28}$$

$$\beta_k = \frac{\sum_{i=1}^{n} p_k^2(x_i)}{\sum_{i=1}^{n} p_{k-1}^2(x_i)} \tag{29}$$

$$b_k^{(m)} = \omega_k / d_{kk} \tag{30}$$

$$\omega_k = \sum_{i=1}^{n} \overline{f}_i\, p_k(x_i) \tag{31}$$

$$d_{jk} = \sum_{i=1}^{n} p_k(x_i)\, p_j(x_i) \tag{32}$$

The method described above was used for approximation of the scale of length *m*

$$m_p \approx \sum_{j=0}^{m} b_j\, p_j(\varphi) \tag{33}$$

The best approximation was used for the 6th order of the polynomial. The coefficients of the following values were determined

b_0	9.91084847676317E-0001
b_1	9.76487121105209E-0001
b_2	1.34430545555791E+0000
b_3	-2.01124868590491E-0001
b_4	1.45535915010413E+0000
b_5	-6.04275366658094E-0001
b_6	2.70713619766045E-0001

Table 4 presents the scale values *m* calculated using the formula (20), m_p approximated using the polynomial (33) and differences determined for selected points located within equal distances.

Table 4

φ	M	m_p	$m-m_p$
48°	3.89324451749	3.89324451657	9.2E-10
49°	4.07680006789	4.07680006820	-3.2E-10
50°	4.27431608521	4.27431608391	1.3E-09
51°	4.48724199273	4.48724199380	-1.1E-09
52°	4.71722041101	4.71722041149	-4.8E-10
53°	4.96611890326	4.96611890167	1.6E-09
54°	5.23606797750	5.23606797963	-2.1E-09

Comparing the results listed in Table 4 and in Table 1 it may be seen, that approximation using Chebyshev series allows for obtaining better results. For the same order of the polynomial, the higher accuracy may be obtained. The projection (18) was approximated by series in the form

$$x \approx \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}\,\varphi^i \lambda^j \tag{34}$$

$$y \approx \sum_{i=0}^{n} \sum_{j=0}^{m} a'_{ij} \varphi^i \lambda^j \qquad (35)$$

Coefficients of the series (34) and (35) were determined by means of the least square method. For the series of the 5th order, specified by the formula (34) the following coefficients were obtained

	i=0	1	2	3	4	5
j=0	-34721.3392	77540.3985	–837314.2058	1587654.6307	-364403.5300	238694.1324
1	236292.5121	-9107.5071	1508652.1512	2.317668.1715	-17481214.6179	12100442.5189
2	-570012.9261	-169335.9288	-1041044.0636	-15710381.4201	60604324.4373	-43707700.9780
3	777390.0566	-416647.4515	3615053.4093	13683419.4510	-67896738.9299	52999715.2076
4	-549098.8848	1008279.7727	-6100457.1573	2890163.0379	25231807.2411	-24177358.1999
5	161890.0986	-491128.5868	2846696.6897	-4775919.0116	-81783.5800	2539596.8107

Table 5 presents the values of the x_p co-ordinate, calculated by means of the series (34) and errors determined at selected points, located within equal distances.

Table 5

φ[°]	λ[°]	x[m]	x_p[m]	x-x_p[m]
48	14	16104.0199	16104.0196	0.0003
52	14	17953.1174	17953.1181	-0.0007
54	14	19025.4827	19025.4815	0.0012
48	18	15784.7063	15784.7058	0.0005
52	18	17597.1396	17597.1403	-0.0007
54	18	18648.2418	18648.2408	0.0010
48	22	15388.4912	15388.4909	0.0003
52	22	17155.4302	17155.4307	-0.0005
54	22	18180.1485	18180.1478	0.0007

For the series (35) of the 5th order the following coefficients were obtained

	i=0	1	2	3	4	5
j=0	-5551.5774	-24503.9660	17723.9481	-78562.1206	-154909.6266	295523.8492
1	38009.0608	139292.1304	-36954.9049	1255853.9791	-1965634.4540	1000471.6324
2	-103517.8703	-166440.3271	–376468.0774	–3617704.6852	9080180.2243	–7129674.9049
3	139541.9708	13066.0505	1278287.8342	3823382.8292	-13593018.8557	12301887.0255
4	-92924.3062	108548.5038	–1376931.8909	–1411392.7509	8550513.8820	–8677232.9696
5	24445.0981	–48282.9616	494538.7801	24253.2983	–1916367.8098	2208788.3584

Table 6 presents the values of co-ordinates, calculated using the formula (19) and the series (35), as well as differences between those values.

Table 6

φ[°]	λ[°]	y[m]	y_p[m]	y-y_p[m]
48	14	4015.1831	4015.1831	0.0000
52	14	4476.2149	4476.2151	-0.0002
54	14	4743.5856	4743.5854	0.0002
48	18	5128.7620	5128.7620	0.0000
52	18	5717.6572	5717.6575	-0.0003
54	18	6059.1811	6059.1808	0.0003
48	22	6217.3540	6217.3540	0.0000
52	22	6931.2437	6931.2439	-0.0002
54	22	7345.2568	7345.2565	0.0003

Comparing the results listed in Tables 5,6 and in Tables 3,4 it may be seen that approximation using Chebyshev series allows for obtaining the higher accuracy. For the same order of the polynomial the higher accuracy may be obtained.

6. Utilisation of Chebyshev polynomials for minimizing distortions of cartographic projections

Basing on works performed by Gdowski (Gdowski 1969), an example of construction of a cartographic projection of an ellipsoid into a plane, for which Chebyshev polynomials will be used for minimizing projection distortions, will be presented in this section. That projection will be compared to the projection determined basing on the method applied by Tissot.

The co-ordinate system is constructed on the ellipsoid in the following way. An arbitrary point *O* is assumed on the ellipsoid and geodetic lines are constructed in each direction. Then, orthogonal trajectories of those lines are constructed, which are the Gauss geodetic circles. Thus, the semi-geodetic grid is obtained on the ellipsoid. Then it is assumed that lines *v*=const are geodetic lines, which cross the point *O,* provided, that *v* is such an angle, which is created by the given geodetic line with the meridian $L = L_O$ of the geodetic co-ordinates (*B,L*). Geodetic circles are marked as *u*=const. If u means the length of the arc of given geodetic lines, then the I square form of the ellipsoid surface may be presented in the form

$$ds^2 = du^2 G(u,v) dv^2 . \tag{36}$$

The $\sqrt{G}$ is presented in the form

$$\sqrt{G} = u - \frac{K}{6}u^3 - \frac{1}{12}K_u u^4 + ..., \tag{37}$$

where *K* means the Gauss curvature of the given surface. Then we assign

$$b = b(v) = -\frac{1}{12}K_u . \tag{38}$$

Now the generalised azimuthal projection should be considered in the form

$$x = r(u)\cos v \tag{39}$$

$y = r(u)\sin v$

where $u \in \langle 0, \gamma \rangle$ and $v \in \langle 0, 2\pi \rangle$.

Conformal projections do not exist in the class of projections (39). However, such projections may be determined out of those projections, which will have the possibly lowest distortion of areas.

Assuming

$$r(u) = \sum_{k=1}^{n} a_k u^k \tag{40}$$

the area deformatin may be presented in the form

$$P = mn - 1 = \frac{rr_u}{\sqrt{G}} - 1 = a_1^2 - 1 + 3a_1 a_2 u + \left(4a_1 a_3 + 2a_2^2 + \frac{K}{6}a_1^2\right)u^2 + \tag{41}$$

$$+\left(5a_1 a_4 + 5a_2 a_3 - ba_1^2 + \frac{K}{2}a_1 a_2\right)u^3 + ...$$

After equating the coefficient of the series (41) to zero, the following values are obtained a_1=1, a_2=0, $a_3 = -\frac{K}{24}$. The coefficient of u^3 is equal to $5a_4$-*b* and it cannot be equated to zero, since a_4 is the constant number, and *b*=*b*(*v*).

Assuming $a_4 = \frac{1}{5}b_0$, where $b_0 = \frac{1}{2\pi}\int_0^{2\pi} b dv$ one obtains

$$p = mn - 1 = (b_0 - b)u^3 + \dots \tag{42}$$

The above presented method was applied in the Tissot theory.
Now, in order to minimize the distortion (42) Chebyshev polynomials will be applied.
Assuming $A = 5a_1b_0 + 5a_2a_3 - ba_1^{\,2} + \frac{K}{2}a_1a_2$ the distortion (42) may be presented in the following form

$$\frac{mn-1}{A} = \frac{a_1^{\,2}-1}{A} + \frac{3a_1a_2}{A}u + \frac{4a_1a_3 + 2a_2^{\,2} + \frac{K}{6}a_1^{\,2}}{A}u^2 + u^3 + \dots \tag{43}$$

Limiting considerations to four terms of the series (43), the issue of minimization of that expression is the issue of determination of such polynomial of the 3rd order of the coefficient at the highest power equal to 1, which has the smallest deviation from 0 in the interval $u \in \langle 0, \gamma \rangle$.
Following the minimax rule, those conditions are met by Chebyshev polynomial. For the interval $x \in \langle -1, 1 \rangle$ it has the form

$$T_3 = x^3 - \frac{3}{4}x, \tag{44}$$

And

$$\max_{x\langle -1,1\rangle}\left| x^3 - \frac{3}{4}x \right| = \frac{1}{4}. \tag{45}$$

In order to determine Chebyshev polynomial of the third order for the interval $u \in \langle 0, \gamma \rangle$ the variable should be modified using the relation

$$x = \frac{2u - \gamma}{\gamma} \tag{46}$$

Then, Chebyshev polynomial of the third order, for the interval $u \in \langle 0, \gamma \rangle$ is obtained in the form

$$\hat{T}_3 = u^3 - \frac{3}{2}\gamma u^2 + \frac{9}{16}\gamma^2 u - \frac{1}{32}\gamma^3 \tag{47}$$

where

$$\max_{u \in \langle 0,\gamma \rangle}\left|\hat{T}_3\right| = \frac{\gamma^3}{32} \tag{48}$$

After equating coefficients in (47) and (43) and solving the system of equations the following relations are obtained

$$a_1 = 1 - \frac{1}{64}(b_0 - b)\gamma^3 + \dots$$

$$a_2 = \frac{3}{16}(b_0 - b)\gamma^2 + \dots \tag{49}$$

$$a_3 = -\frac{K}{24} - \frac{3}{8}(b_0 - b)\gamma + \dots$$

Thus, the projection, for which distortion of areas is approximately equal

$$p = mn - 1 \approx (b_0 - b)\hat{T}_3 \tag{50}$$

was determined, with the maximum deviation from zero, equal

$$\frac{\gamma^3}{32}(b_0 - b). \tag{51}$$

The maximum distortion (51) is approximately 32 times smaller than the distortion (42).

Concluding remarks

The method of the uniform approximation, described in this paper, is the alternative for the commonly used method of the mean square approximation. Using series based on orthogonal Chebyshev polynomials the necessity to solve large systems of equations may be avoided, what allows to avoid many numerical problems. As it may be seen on the presented example, the described method gives better results with respect to the accuracy, than typical methods applied in the case of the mean square approximation. The simple construction of algorithms used for calculations, is the additional advantage, which convinces to the selection of this method.

The minimax rule allows for utilisation of Chebyshev polynomials for construction of cartographic projections, and allows for minimizing the projection distortions.

References

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