CO-301

SELECTED ISSUES OF UTILISATION OF EQUI-AREA MAP PROJECTIONS FOR CALCULATION OF AREAS OF ADMINISTRATIVE UNITS IN POLAND

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Calculation of areas of polygons is one of the basic tasks performed in geodetic and cartographic works. That issue appears in many tasks. For example, they may be connected with maintaining a real estate cadastre or with calculation of areas of administrative units. Such works have become particularly important when new geocentric ellipsoids WGS84 and GRS80 are used. Calculation of a polygon area on the ellipsoid becomes a complicated task, since precise formulae, which would allow for implementation of that task do not exist; only approximate formulae, which may be applied for small areas, are known. Therefore it is difficult to perform calculations in case of large polygons located on a flattened ellipsoid of revolution, which requires utilisation of non-standard algorithms.

One of the possibilities is to use cartographic, equi-area projections. Transformation of co-ordinates of vertices of polygons, from a plane of practically applied co-ordinate systems to the plane of the equi-area projection, allows for calculations of areas of ellipsoidal polygons directly on a plane of the equi-area projection.

The issue of ambiguity of results of such calculations arises. Different values of polygon areas, determined by the same vertices are obtained in various projections. In order to achieve the compliance of calculated areas, the figure must be explicitly defined in its original form (for example, the, so-called, geodetic figure may be assumed, i.e. such a figure, the side of which are sections of geodetic lines) and areas of curvilinear equivalents of those figures should be calculated in the projection plane. Those difficulties may be minimised by searching for cartographic projection of the possibly smallest projection deformations.

The paper will analyse influence of linear projection deformations on the value of calculated areas of polygons in cartographic equi-area projections. Results of analyses for typical projection criteria, widely applied in cartography and in geographic information systems, as well as criteria of minimisation of deformations, applied within the given area, will be presented.

Besides, algorithms for calculations of areas of curvilinear image equivalents of ellipsoidal figures will be presented and some problems of using them will be explained.

Results of utilisation of developed algorithms for calculation of areas of administrative units in Poland will be also discussed.

1. GENERAL PROPERTIES OF EQUI-AREA PROJECTIONS

Equi-area projections are characterised by numerous interesting metric properties. The basic property of those projections is maintenance of areas, i.e. the areas of original figures are equal to areas of topological equivalents (imagery) of those figures in the projection plane.

Another property of equi-area projections concerns the fact that the extreme scales of length deformation are mutual inverses in each point of the image plane

$$m = \frac{1}{n}$$

where m,n are extreme scales of length deformations in principal directions.

Therefore, when investigating the distribution of length deformations, we will limit our research to determination of one of the extreme scales.

The maximum angular deformation may be determined using the formula

$$\omega_{\max} = 2\arctan\left(\frac{1}{2}(m-n)\right)$$

2. ANALYSIS OF DISTRIBUTION OF PROJECTION DEFORMATIONS IN SELECTED EQUI-AREA PROJECTIONS

In this section results of analysis of projection deformations for selected equi-area projections will be presented. Both, typical cartographic projections, widely applied in geographic information systems, as well as cartographic projections, developed basing on criteria of minimisation of projection deformations, will be presented. Appropriate projections were determined for the area of Poland and the Mazovia province.

2.1. Cylindrical projections of an ellipsoid into a plane

Projection length deformations in such projection are relatively high. The maximum length deformations for Poland equal to about 70 m/km in southern and northern Poland. For the Mazovia province length deformations equal to 30 m/km and angular deformations equal to 3.5°.

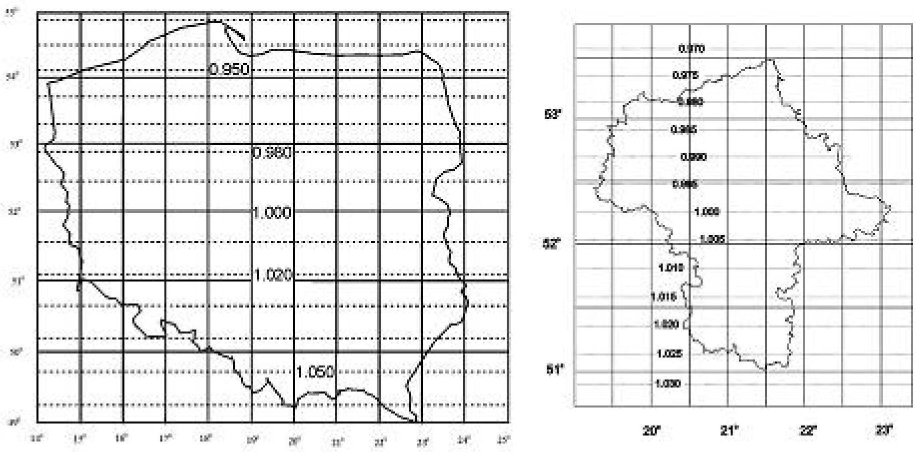

Fig. 1 Isoscales in the cylindrical, equi-area projection of Poland and the Mazovia province.

2.2. Normal azimuthal projections of an ellipsoid into a plane

Projection length deformations in this projection are relatively smaller than in the case of the cylindrical projection. For the area of Poland, the maximum length deformations equal to about 67 m/km in southern and northern Poland. The maximum angular deformations equal to about 7.4°. For the Mazovia province the length deformations equal to 57 m/km and angular deformations reach about 6.8°.

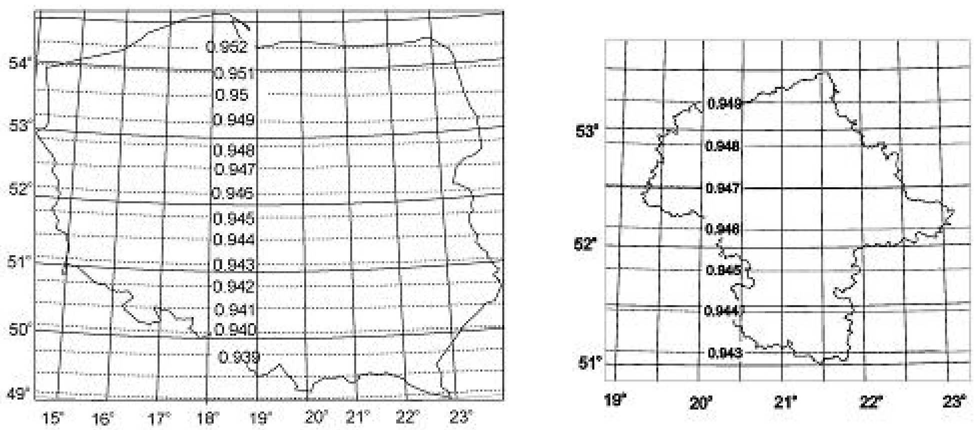

Fig. 2 Isoscales of length in the azimuthal, equi-area projection of Poland and the Mazovia province.

2.3. The sinusoidal Sanson projection of al ellipsoid into a plane

In this projection length projection deformations equal to 1/2 of deformations which occur on the cylindrical projection. The maximum length deformations for Poland equal to about 35 m/km in southern and northern Poland. The maximum angular deformations equal to about 4°. For the Mazovia province the length deformations equal to 14 m/km and angular deformations reach the value of 1.6°.

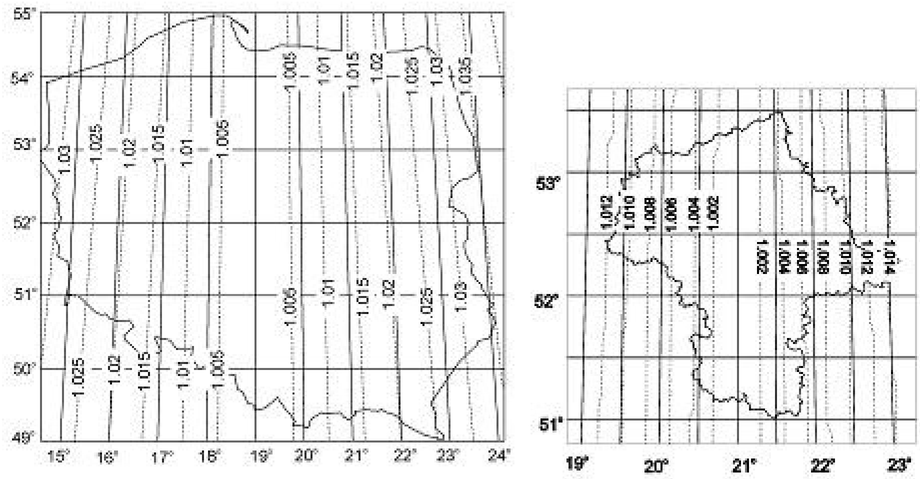

Fig. 3 Isoscales in sinusoidal, equi-area projection for Poland and the Mazovia province

2.4. The Bonne projection

In this projection, length projection deformations are much smaller than in the above presented projections. For the area of Poland the maximum deformations equal to about 1 m/km. The maximum angular deformations equal to about 6'. For the Mazovia province the length deformations equal to 10 cm/km and angular deformations reach the value of 41.2".

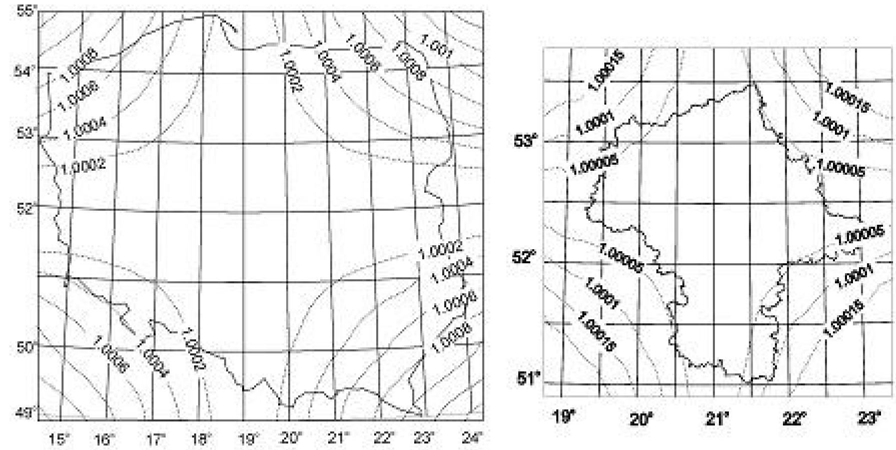

Fig. 4 Isoscales of length in the Bone projection for Poland and the Mazovia province

2.5. Conical projections determined using the Euler criterion

In conical projections, which have been determined using the Euler criteria, the isoscales have the form of circular arcs, parallel to images of parallels. The maximum deformations of length occur in southern and northern Poland and equal to about + 60 cm/km. The maximum angular deformations equal to about 4'. For the Mazovia province the maximum length deformations equal to 10 cm/km and deformations of angles reach the value of 41.2".

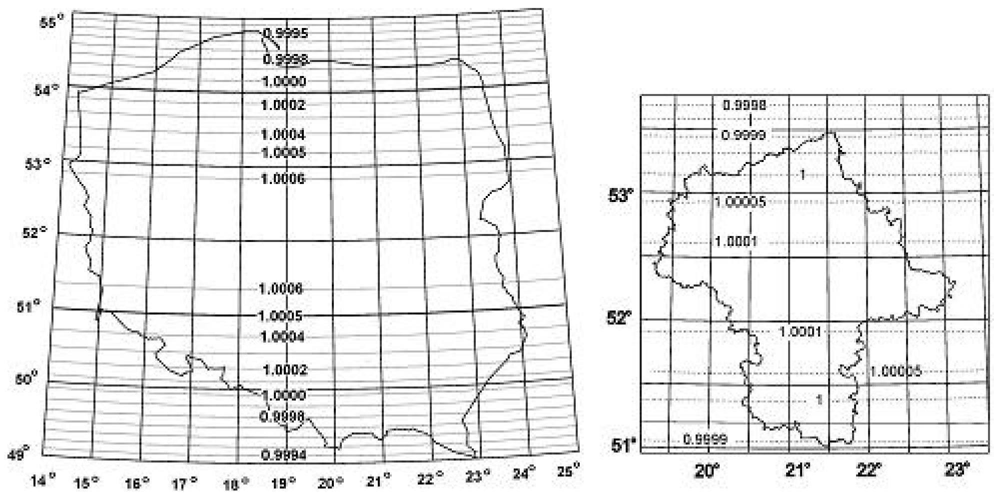

Fig. 5 Isoscales of length in the conical Euler projection of Poland and the Mazovia province

3. ANALYSIS OF INFLUENCE OF PROJECTION DEFORMATIONS ON THE VALUES OF DETERMINED AREAS OF POLYGONS

Areas of polygons have been calculated for projections, which are presented in the previous section. The polygons were created basing on co-ordinates of points, located on the Polish frontiers. Calculations have been performed for 5 polygons, for which the mean distances between vertices equalled to 9 km (max. 47 km), 2 km (max. 12 km), 1 km (max. 6 km), 0.5 km (max. 3 km) and 0.3 km (max. 1.5 km), respectively. Table 1 presents results of areas of polygons in particular equi-area projections and in the "1992" co-ordinate system, applied in Poland and based on the Gauss-Krűger projection.

Such calculations have been also performed for a polygon formed basing on points located on borders of the Mazovia province. The mean distances between points equaled to 0.5 km (max. 4.5 km). Table 2 presents results of calculations of areas of polygons in particular equi-area projections and in the „1992" system.

Several regularities may be noticed basing on obtained results of calculations. In case of longer distances between vertices, higher discrepancies between calculated values of areas occur. Besides, the smaller length deformations, the smaller the differences. We may notice that, in the case of projections, which are characterised by small deformations, i.e. the Bonne and Euler projections, those differences equal to several dozens of square metres only.

Table. 1 Results of calculations of areas of polygons for selected cartographic projections of Poland

Projection	Length deformations	Area 1[m2]	Area 2[m2]	Area 3[m2]	Area 4[m2]	Area 5[m2]
Cylindrical	70 m/km	312 554 066 331	312 556 115 161	312 556 217 604	312 556 243 215	312 556 249 618
Azimuthal	67 m/km	312 557 210 821	312 556 311 694	312 556 266 737	312 556 255 498	312 556 252 689
Sanson	35 m/km	312 550 778 174	312 555 909 656	312 556 166 228	312 556 230 371	312 556 246 407
Bonne	1m/km	312 555 647 854	312 556 214 009	312 556 242 316	312 556 249 393	312 556 251 163
Euler	60 cm/km	312 555 691 970	312 556 216 766	312 556 243 005	312 556 249 565	312 556 251 206
"1992" System	70 cm/km	312 351 252 481	312 351 602 881	312 351 620 401	312 351 624 781	312 351 625 877

Table. 2 Results of calculation of areas of polygons for selected cartographic projections for the Mazovia province

Projection	Length deformations	Area[m2]
Cylindrical	70 m/km	35 586 539 638
Azimuthal	57 m/km	35 586 533 833

Sanson	14 m/km	35 586 536 254
Bonne	10 cm/km	35 586 534 858
Euler	10 cm/km	35 586 534 880
"1992" System	70 cm/km	35 557 049 616

Therefore, in the case of equi-area projections we may obtain high differences (Tables 1 and 2) between calculated sizes of polygons based on the same vertices. However, we may obtain the higher accuracy of calculated areas than in case of conformal projections, e.g. in the "1992" System.

Those differences result from the fact that the topological equivalent of the straight line section, determined in the plane of a given projection, is the arc of a curve line in the plane of another cartographic projection, with the obvious exception of certain particular cases. Therefore, using the equi-area projections for calculation of sizes of polygons, those polygons should be explicitly defined on the original surface and their topological equivalents should be determined in the image plane.

In surveying and cartographic works we deal with polygons, which sides are geodetic lines. In cartographic projections the images of geodetic lines are certain curve lines, not the straight lines. Thus, using equi-area projections for calculating areas of those polygons, we should remember that the area of a geodetic ellipsoidal polygon is maintained, but a certain curvilinear polygon is its topological equivalent. In order to achieve the compliance of calculated areas between the original and the image in cartographic projections, in the projection plane, the area of that curvilinear polygon should be integrated. So, it is necessary to consider reductions of areas. That issue may be minimised by minimising distances between the vertices and by selection of projection parameters, in order to minimise projection deformations, and, therefore, the values of reduction of areas.

4. DETERMINATION OF REDUCTION OF AREAS IN EQUI-AREA PROJECTIONS

If the geodetic polygon is determined on the original surface in the cartographic projection, i.e. the polygon, which sides are sections of geodetic lines, its image in the projection plane will not be – in general – a geodetic figure, but a curvilinear polygon, which sides are not sections of straight lines, but they are arcs of curve lines. Therefore, for each geodetic figure, defined on the original surface, may be assigned a figure in the image plane; that figure is called the reduction equivalent. The reduction equivalent is constructed of sections of straight lines, which connect image equivalents of vertices of the original figure.

Differences or quotients occurring between corresponding metric parameters of the geodetic figure, located on the original surface and the reduction equivalent of that figure in the image plane, are called the projection geodetic reductions. Projection geodetic reductions concern the lengths of sides, internal angles or azimuths of sides, as well as areas of geodetic figures.

If - as the original surface in the cartographic projection - the rotating, flattened ellipsoid of the following equation, is assumed

$$\vec{r} = \vec{r}(B, L) = \left[\frac{a\cos B\cos L}{\sqrt{1 - e^2\sin^2 B}}, \frac{a\cos B\sin L}{\sqrt{1 - e^2\sin^2 B}}, \frac{a(1 - e^2)\sin B}{\sqrt{1 - e^2\sin^2 B}} \right]$$

and the image

$$\vec{r}' = \vec{r}'(B,L) = [x = x(B,L),\ y = y(B,L)]$$

In the plane z=const and Δs will be the length of the geodetic line section, passing through the points P1 and P2 located on the original surface and A1 will be azimuth of this line at the point P1, and A2 will be the azimuth of the geodetic line at the point P2 (Fig.6), then the image of the section of the geodetic line P1P2 in the cartographic projection will be the curvilinear section P1'P2' of the length Δs', the image of the azimuth A1, will be the direction angle A1' between the vector r_B' , tangential to the image of the meridian at the point P1, and the vector dr'/dL tangential to the image of the geodetic line at that point. On the other hand, the image of the azimuth A2, will be the direction angle A2', calculated with the reverse sign, between the vector r_B' tangential to the image of the meridian at the point P2, and the vector *dr'/dL* tangential to the image of the geodetic line at that point, as it has been presented in Fig. 6.

The reduced equivalent of the geodetic line P1P2 will be the section of the straight line, connecting points P1’ and P2’ of the length Δs" ; the reduced equivalent of the direction angle A1 will be the direction angle between the vector tangential to the image of the meridian at the point P1’ and the section P1”P2”; the reduced equivalent of the direction angle A2, will be the direction angle between the vector r_B' , tangential to the image of the meridian at the point P2’, and the section P1”P2”. Angles $\delta_{12}=A_1''-A_1'$, $\delta_{21}=A_2''-A_2'$ are called the reduction angles (Fig. 6).

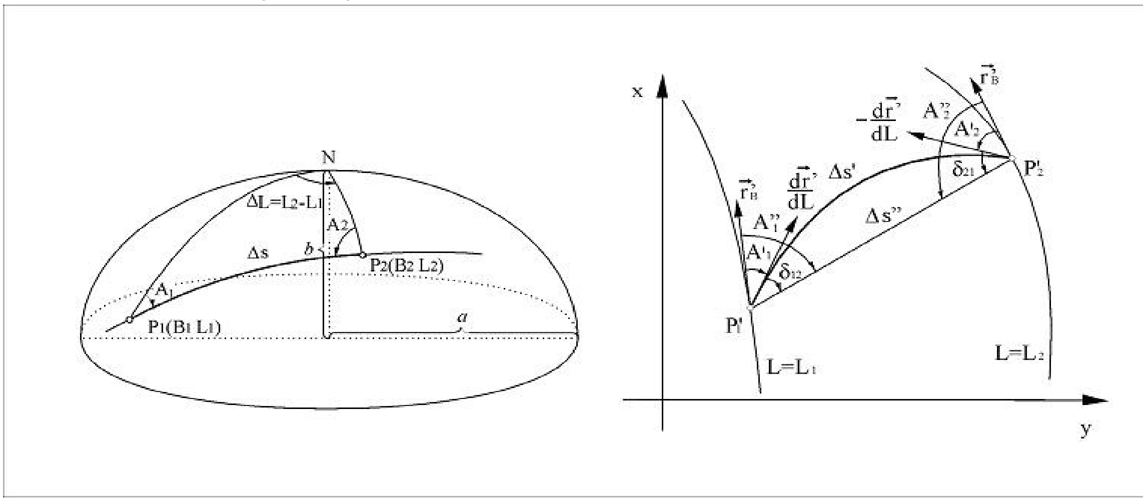

Fig. 6. Topological and reduction equivalents of a geodetic line

The projection reduction of the area Δ’F (Fig. 6) constructed on the chord P1’ P2’ is expressed by the approximate formula

$$\Delta' F = \frac{\Delta s''^2}{4}\left(\frac{\delta - 0.5\sin 2\delta}{\sin^2 \delta}\right)$$

where Δs” – the chord length P1’ P2’ and δ=0.5(δ12+δ21).

After consideration of the reduction value of the area constructed on the chord of the image of the geodetic line, allows for accurate computing of areas of geodetic polygons . Tables 3 and 4 present results of calculations of areas of polygons defined in section 3.

Table. 3 Results of calculations of areas of polygons for selected cartographic projections of Poland, after consideration the area reduction

Projection	Length deformations	Area 1[m2]	Area 2[m2]	Area 3[m2]	Area 4[m2]	Area 5[m2]
Cylindrical	70 m/km	312 556 251 772	312 556 251 752	312 556 251 751.8	312 556 251 751.8	312 556 251 752.4
Azimuthal	67 m/km	312 557 251 778	312 556 251 752	312 556 251 751.8	312 556 251 751.9	312 556 251 752.4
Sanson	35 m/km	312 556 251 708	312 555 251 752	312 556 251 751.7	312 556 251 751.9	312 556 251 752.4
Bonne	1m/km	312 556 251 803	312 556 251 754	312 556 251 752.3	312 556 251 752.0	312 556 251 752.4
Euler	60 cm/km	312 556 252 056	312 556 251 770	312 556 251 756.3	312 556 251 753.0	312 556 251 752.7

Table. 4 Results of calculations of areas of polygons for selected cartographic projections of the Mazovia province, after consideration the area reduction

Projection	Length deformations	Area[m2]
Cylindrical	70 m/km	35 586 535 046.0
Azimuthal	57 m/km	35 586 535 045.6
Sanson	14 m/km	35 586 535 046.0

Bonne	10 cm/km	35 586 535 047.9
Euler	10 cm/km	35 586 535 045.5

When analyzing obtained results we may notice the high compliance between calculated areas. It is obvious that the higher compliance of results is obtained when the distance between points is decreased. Application of the algorithm of calculation of projection reductions, based on approximation of the image of the geodetic line by the circular arc produces good results for small distances between points; thus, differences between calculated areas are bigger for longer distances.

In the case of long distances between points, other methods should be used for calculation of reduction of areas, based on numerical integration, such as the Simpson method. Then we determine rectangular, plane co-ordinates for the even division of the section of the image of the geodetic line P0Pn. Having the rectangular co-ordinates of ends of the section (x0,y0) and (xn,yn) we perform transformation of the co-ordinates of points of division of the section of the geodetic line to the local system x'oy', the origin of which is the point P0 and points P0 and Pn determine the direction of the y' axis of the new system.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right)$$

Then we calculate the area Δ'F, using the integral Simpson formula

$$\Delta F' = \frac{h}{3}\left[x'_0 + 4\left(x'_1 + x'_3 + x'_5 + \ldots + x'_{n-1} \right) + 2\left(x'_2 + x'_4 + x'_6 + \ldots + x'_{n-2} \right) -\right.$$

where

$$h = \frac{y'_n - y'_0}{n}$$

Calculation of plane, rectangular coordinates of points located in the image of the section of the geodetic line, is performed in two following stages:

1. Using the methods of transfer of co-ordinates along the geodetic line on the ellipsoid, we calculate co-ordinates of points of even division of that section,

2. We calculate co-ordinates of images of those points in the plane of the equi-area projection.

When calculating reductions of areas, we may apply the equation of the image of the geodetic line in the projection plane. Below the equation of the image of the geodetic line in the Sanson projection plane is presented.

$$x = \int_{B0}^{B} M\,dB = \int_{B_0}^{B} \frac{a\left(1-e^2\right)dB}{\sqrt{\left(1-e^2\sin^2 B\right)^3}}$$

$$y = N\cos B = \frac{aL\cos B}{\sqrt{1-e^2\sin^2 B}}$$

Where M,N are radii of basic curvatures of the ellipsoid.

The equation fo the geodetic line on the ellipsoid may be presented in the following form (Gdowski 1968)

$$L - L_1 = he'^2 \int_{B_1}^{B} \frac{dB}{\cos B\sqrt{1-e^2\sin^2 B}\sqrt{\left(a^2-h^2\right)-\left(a^2-h^2e^2\right)\sin^2 B}}$$

where h is the Clairauta constant for the geodetic line

$$h = \frac{a\cos B_1 \sin \alpha_1}{\sqrt{1 - e^2 \sin^2 B_1}}$$

where α_1 is the azimuth of the geodetic line at the point P1 (B1, L1).
After substitutions the equation of the image of the geodetic line in the projection plane will have the form

$$x = \int_{B_0}^{B} \frac{a(1 - e^2)dB}{\sqrt{(1 - e^2 \sin^2 B)^3}}$$

$$y = \frac{a\cos B}{\sqrt{1 - e^2 \sin^2 B}} \left(L_1 + he'^2 \int_{B_1}^{B} \frac{dB}{\cos B\sqrt{1 - e^2 \sin^2 B}\sqrt{(a^2 - h^2) - (a^2 - h^2 e}} \right.$$

5. UTILISATION OF DEVELOPED ALGORITHMS

In Poland the State Register of Borders (PRG) is responsible for registering, collection and distribution of information concerning the territorial division units, their borders and areas. Data concerning borders origins from the registers of lands and buildings. In the PRG co-ordinates of points in the reference, geodetic co-ordinate system GRS80 and in the 1992 System. The administrative area of Poland, including areas of administrative units of the three-level administrative division, are calculated basing on co-ordinates of border points (B, L) in the 1992 System, using corrections of areas from the projection plane onto the GRS-80 ellipsoid plane. The algorithm is based on the division of areas in the plane of the "1992" projection system, into elementary figures, and then, on summing them with consideration of influences of scales of areas distortion. This algorithm does not consider values of projection reductions.

In this paper, the alternative solution is proposed, which comprises transformation of co-ordinates of points, which delineate borders of administrative units from the „1992" System to the equi-area projection, with appropriate selection of parameters, which allows for minimisation of projection deformations and then, calculation of the area of the plane polygon. In order to achieve higher accuracy, reduction of area may be possibly considered.

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