CO-324

SOME PECULIARITIES OF GENERATING PROJECTIONS FOR A TRIAXIAL ELLIPSOID

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Creating a mathematical basis for maps of any object implies a transition from its physical surface to a reference surface that can be described by equations; then the mathematical surface is represented on a plane. Traditionally in mathematical cartography, a sphere and an ellipsoid of revolution are used as reference surfaces. [1] analyses three approaches to mapping celestial bodies whose shapes differ greatly from a sphere or an ellipsoid of revolution. They are: an application of a morphographic projection, a representation of separate sides of a celestial body, and usage of more complicated reference surfaces (triaxial or multiaxial ellipsoids).

The problems of creating projections for a triaxial ellipsoid are considered in the works by Bugaevsky L.M. [2,3].

A lot of curvilinear coordinate systems can be set on the surface of a triaxial ellipsoid and so on any regular surface. [2] deals with various coordinate systems, including geodetic and pseudo geodetic ones. It is possible to apply any one, taking into consideration only the relative simplicity of its mathematical apparatus, as the properties of a projection do not depend on the co-ordinate system used.

For a triaxial ellipsoid with semi-axes a, b, c, we will use a pseudo geodetic co-ordinate system considered in [2]. By a pseudo-geodetic longitude λ we will mean a dihedral angle between the sectional planes passing through an axis of an ellipsoid. Thus, pseudo-geodetic and geodetic longitudes coincide. Meridians are the lines of cutting by these planes on the surface of a triaxial ellipsoid and look like ellipses with semi-axes c and d. The semi-axis c is a constant; the semi-axis d depends on λ and is determined by formula

$$d = \frac{b}{(1-k^2\cos^2\lambda)^{1/2}}, \qquad (1)$$

where $k^2 = 1-\left(\frac{b}{a}\right)^2$.

The radius of meridian section is determined by formula

$$M = \frac{d(1-p^2)}{(1-p^2\sin^2 B)^{3/2}}, \qquad (2)$$

where

$$p^2 = 1-\left(\frac{c}{d}\right)^2.$$

The pseudo-geodetic latitude of a point is an angle between the normal to the meridian line (i.e. to an ellipse) at a point A and the equatorial plane. As the normal to the meridian is not the normal to the ellipsoid surface, geodetic and pseudo-geodetic latitudes do not coincide. By a parallel we will understand a line with equal pseudo-geodetic latitudes *B=const*.

The grid of pseudo-geodetic coordinates will be not orthogonal. The angle of divergence from the right one between meridians and parallels is determined by formulas [2]

$$tg\,c = -\frac{z\sin B}{(1+z^2\cos^2 B)^{1/2}}, \qquad (3)$$

where

$$z = -\frac{d_\lambda}{d} = \frac{k^2\sin 2\lambda}{2(1-k^2\cos^2\lambda)} \qquad (4)$$

As the grid of meridians and parallels is not orthogonal, it is necessary to complete construction of an infinitesimal spheroidal trapezoid formed by meridians and parallels, to make a rectangle in order to determine the length of a linear element on the surface of a triaxial ellipsoid. This can be made in two ways.

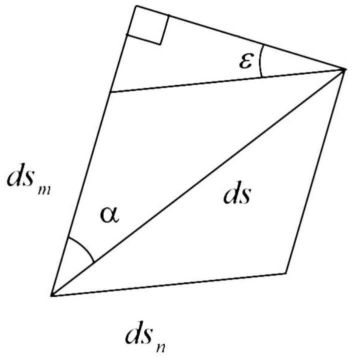

Fig. 1

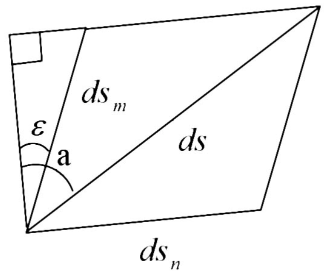

Fig. 2

In the figures,

$ds_m = MdB$ - is an infinitesimal segment of a meridian;

$ds_n = r(1+z^2)^{1/2}d\lambda$ - is an infinitesimal segment of a parallel

$$r = N\cos B$$
$$N = \frac{a}{\sqrt{1-e^2\sin^2 B}} \qquad (5)$$

In the first case (Fig. 1), the length of a linear element and the angle are determined as

$$ds^2 = \left[MdB + r(1+z^2)^{1/2}\sin\varepsilon d\lambda\right]^2 + \left[r(1+z^2)^{1/2}\cos\varepsilon d\lambda\right]^2 \qquad (6)$$

$$tg\alpha = \frac{r(1-z^2)^{1/2}\cos\varepsilon d\lambda}{MdB + r(1+z^2)^{1/2}\sin\varepsilon d\lambda}, \qquad (7)$$

where α - is the azimuth of the linear element measured from a meridian.

In the second case (Fig. 2), the length of the linear element and the angle are determined as

$$ds^2 = \left[M\sin\varepsilon dB + r(1+z^2)^{1/2}d\lambda\right]^2 + \left[M\cos\varepsilon dB\right]^2 \qquad (8)$$

$$tg\mathrm{a} = \frac{r(1+z^2)^{1/2}\cos\varepsilon d\lambda}{MdB - r(1+z^2)^{1/2}\sin\varepsilon d\lambda}, \qquad (9)$$

where a – is the angle measured from a direction that is orthogonal to a parallel.

The formula of the length of a linear element on a flat surface is known from the general theory of cartographic projections. It is

$$d\sigma^2 = edB^2 + 2fdBd\lambda + gd\lambda^2, \qquad (10)$$

where *e, g, f, h* – are the Gaussian coefficients.

In view of formulas (6) and (8), the general formula of the local length scale can be also rewritten in different ways:

$$\mu^2 = m^2(\cos\alpha - tg\varepsilon\sin\alpha)^2 + \frac{mn\cos i(\sin 2\alpha - 2tg\varepsilon\sin^2\alpha)}{\cos\varepsilon} + \frac{n^2\sin^2\alpha}{\cos^2\varepsilon} \qquad (11)$$

or

$$\mu^2 = \frac{m^2 \cos^2 a}{\cos^2 \varepsilon} + \frac{mn \cos ii(\sin 2a - 2tg\varepsilon \cos^2 a)}{\cos \varepsilon} + n^2(\sin a - tg\varepsilon \cos a)^2 \qquad (12)$$

where *ii* – is the angle between a meridian and a parallel on a flat surface, it is defined as

$$tg ii = \frac{h}{f} \qquad (13)$$

m, n – are the scales along meridians and parallels, defined by the formulae

$$m = \frac{\sqrt{e}}{M}$$
$$n = \frac{\sqrt{g}}{r\sqrt{1+\varepsilon^2}} \qquad (14)$$

The solution of the equations of the forms $\frac{\partial \mu^2}{\partial \alpha} = 0$ and $\frac{\partial \mu^2}{\partial a} = 0$ allows one to determine the azimuths of the principal directions.

$$tg 2\alpha = \frac{m^2 \sin 2\varepsilon - 2mn \cos ii \cos \varepsilon}{n^2 - 2mn \cos ii \sin \varepsilon - m^2 \cos 2\varepsilon} \qquad (15)$$

and

$$tg 2a = \frac{n^2 \sin 2\varepsilon - 2mn \cos ii \cos \varepsilon}{n^2 \cos 2\varepsilon - 2mn \cos ii \sin \varepsilon - m^2} \qquad (16)$$

The angles α , α + 90 and a, a + 90 calculated by formulae (15) and (16) differ by the value of ε

α = a - ε

After substituting the calculated values of α and a into formulae (11) and (12), respectively, we will get the values of the extreme length scales.

The extreme length scales a and b can be also derived by Apollonius' theorems. However, Apollonius' theorems are valid only for conjugate semi-diameters of an ellipse. Conjugate on a flat surface become only those directions that are orthogonal on an ellipsoid. As the grid of meridians and parallels of a triaxial ellipsoid is not orthogonal, these directions are not conjugate, which makes it impossible to apply Apollonius' theorems of the traditional form.

To apply Apollonius' theorems it is necessary to find both n_ε - the length scale along the direction that is conjugate to a meridian and τ - the angle between the meridian and this direction in the projection. In this case Apollonius' theorems will have the following forms:

$$A = (m^2 + n_\varepsilon^2 + 2mn_\varepsilon \sin \tau)^{\frac{1}{2}}$$
$$B = (m^2 + n_\varepsilon^2 - 2mn_\varepsilon \sin \tau)^{\frac{1}{2}}$$
$$a = \frac{A+B}{2}$$
$$b = \frac{A-B}{2} \qquad (17)$$

One can find both m_ε - the length scale along the direction conjugate to a parallel and the angle τ' - the angle between it and the parallel.

In this case Apollonius' theorems should be applied in the following form

$$A = (m_\varepsilon^2 - n^2 - 2m_\varepsilon n \sin \tau')^{\frac{1}{2}}$$
$$B = (m_\varepsilon^2 - n^2 - 2m_\varepsilon n \sin \tau')^{\frac{1}{2}}$$
$$a = \frac{A-B}{2}$$
$$b = \frac{A-B}{2} \qquad (18)$$

To determine the angles τ and τ' it is necessary to apply the formula of the relationship between azimuths in the projection and on the ellipsoid.

If this problem is solved with an application of the azimuths α , then

$$\mu^2 = m^2(\cos\alpha - tg\varepsilon \sin\alpha)^2 + \frac{mn\cos ii(\sin 2\alpha - 2tg\varepsilon \sin^2\alpha)}{\cos\varepsilon} + \frac{n^2 \sin^2\alpha}{\cos^2\varepsilon} =$$
$$= \sec^2\varepsilon\left[m^2\cos^2(\alpha+\varepsilon) - 2mn\cos ii \sin\alpha \cos(\alpha-\varepsilon) + n^2\sin^2\alpha\right] \qquad (19)$$

$$ctgA = \frac{m\cos\varepsilon}{n\sin ii}(ctg\alpha - tg\varepsilon) + ctg\,ii \qquad (20)$$

At α =0, we will have the length scale along the meridian and its azimuth
$\mu=m$, $A=0$.
At α =90, we will have the length scale and the azimuth of the direction that is conjugate to the meridian

$$n_c^2 = \sec^2\varepsilon(m^2\sin^2\varepsilon - 2mn\cos ii \sin\varepsilon + n^2)$$
$$ctgA' = -\frac{m\sin\varepsilon}{n\sin ii} + ctg\,ii \qquad (21)$$

The angle between them in the projection is $\tau = A' - A = A'$.
Extreme length scales are calculated by formulae (17).
At α =90-ε, we can find the length scale along the parallel and its azimuth in the projection $\mu=n$, $A=ii$.
At α =-ε we will have the length scale and the azimuth of the direction that is conjugate to the parallel.

$$m_c^2 = \sec^2\varepsilon(m^2 - 2mn\cos ii \sin\varepsilon + n^2\sin^2\varepsilon)$$
$$ctgA'' = -\frac{m}{n\sin\varepsilon \sin ii} + ctg\,ii \qquad (22)$$

The angle between the parallel and the direction that is conjugate to it in the projection is $\tau' = ii - A''$. We can find the values of the extreme length scales by formulae (18).
The same result can be achieved if we set arbitrary values of α and α +90 and compute their azimuths in the projection and their length scales by formulae (19) and (20). These directions will be conjugate, and an application of Apollonius' theorems to them will provide us with the same result, i.e. we will find the values of extreme scales.
The problem with the usage of the angles a, measured from the direction that is orthogonal to parallels is solved similarly.
Another problem considered in the paper is connected with the selection of the shape of the auxiliary surface applied to cylindrical projections. In cylindrical projections of an ellipsoid of revolution or that on a sphere, the circular cylinder is traditionally used as an auxiliary developable surface. The cylinder can be both tangent and secant. In the former case, it is the equator that is represented without any distortions; the section parallel is shown without distortions in the latter case.
If a triaxial ellipsoid with the equator in the shape of an ellipse is used as the surface to be mapped the application of the circular cylinder will result in increased distortions. It is more expedient to project the surface of a triaxial ellipsoid on the cylinder with the ellipse as its base and with the parameters similar to those of the equator. In this case the equator will be tangent to the cylinder and its length will be represented without distortions. Due to that the meridians will not be equidistant straight lines, and the formula for the ordinates will become of the following form:
$y=S$,
where S – is the length of an arc along the equator from the initial meridian up to the current one.
Fig. 3 shows the plane of an ellipsoid equator with the semi-axes a and b.
A - is a current point, λ - is its longitude.
F – is the angle between the normal at the point A and the x- axis.

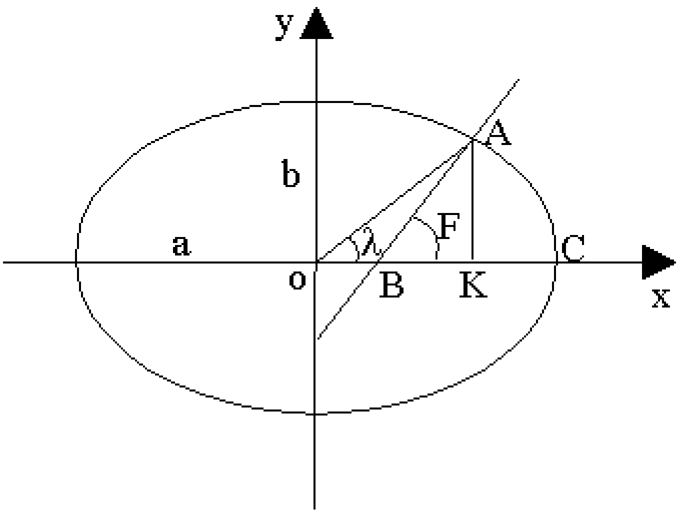

Fig. 3.

The length of an arc of the ellipse AC= S is determined by the known formula

$$S = \frac{a}{1+n}\left[\left(1+\frac{n^2}{4}+\frac{n^4}{64}+\dots\right)F-\left(\frac{3}{2}n-\frac{3}{16}n^3-\dots\right)\sin 2F+\left(\frac{15}{16}n^2-\frac{15}{64}n^4+\dots\right)\sin 4F-\left(\frac{35}{48}n^3+\dots\right)\sin 6F+\dots\right]$$

where $n=\frac{a-b}{a+b}$ (23)

The formula for the angle *F* can be derived from the joint solution of the equator equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and that of the intersecting line of the equator and the meridian section $y=x\,tg\lambda$, as well as with regard to the equations for the normal at the point A. Thus, the formula for relationship of the angles *F* and λ will have the form

$$tgF=\frac{tg\lambda}{(1-k^2)}$$ (24)

Let us compute the coordinates and distortions of one of cylindrical projections to evaluate the efficiency of the cylinder with an elliptical base. As an example we refer to the perspective cylindrical projection for Amalthea generated under the stipulation that the point of view is infinitely distant – the tangent cylinder. The surface of Amalthea is approximated by a triaxial ellipsoid with the semi-axes equal to: a=135 km, b=85 km, c=77.5 km.

The formulae of the projections will take the following forms:

Option 1 (the circular cylinder)

$$x=N(1-p^2)\sin B$$
$$y=a\lambda$$ (25)

Option 2 (the cylinder with an elliptical base)

$$x=N(1-p^2)\sin B$$
$$y=S$$ (26)

The values of the extreme length scales a and *b* have to be invoked by one of the above ways in order to describe the distortion values in the both options of the projections.

The values calculated of the extreme scales a and b allow one to compute both the scale of the area *p*=$a\ b$ and the distortions of the angles $\sin\frac{\omega}{2}=\frac{a\ b}{a-b}$.

An integrated index will be used to determine the distortion level across the whole territory to be mapped.

$$E=\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2$$, (27)

where ε - is a complex criterion at a particular point.

By way of complex criteria there have been applied two Airy criteria computed at the intersection points of meridians and parallels.

$$\varepsilon_1^2 = \frac{1}{2}\left[(a-1)^2 + (b-1)^2\right]$$
$$\varepsilon_2^2 = \frac{1}{2}\left[(ab-1)^2 + \left(\frac{a}{b}-1\right)^2\right] \quad (28)$$

The values of the integrated indexes for the two options of the projections are given in Table 1.

Table 1.

The values of the integrated indexes		
Integrated indexes	Option 1	Option 2
E1	1.186	0.515
E2	20.64	10.86

The values of the integrated indexes bear witness to the advantage of using the cylinder with an elliptical base.

It should be noted that it would be more accurate to determine the integrated parameters with allowance for areas, i.e. by the formula

$$E = \frac{1}{S}\sum_{i=1}^{n}\varepsilon_i^2 \Delta S_i , \quad (29)$$

where ΔS_i - is the areas of the spheroidal trapezoids of a triaxial ellipsoid, their central points having a defined Airy criterion.

S – is the whole territory to be mapped.

The determination of the areas of the spheroidal trapezoids for the triaxial ellipsoid reduces to finding the definite integral of the form

$$S = \int_{\lambda_1}^{\lambda_2}\int_{B_1}^{B_2} Mr\left(1+z^2\right)^{1/2} \cos\varepsilon \, dB \, d\lambda \quad (30)$$

In view of the formulae for M, r, z and ε , it is possible to write

$$S = \int_{\lambda_1}^{\lambda_2}\int_{B_1}^{B_2} \frac{c^2\sqrt{1+z^2\cos^2 B}}{\left(1-p^2\sin^2 B\right)^2} \cos B \, dB \, d\lambda \quad (31)$$

As finding an integral like that is an intricate problem the integration has been done by numerical methods with the use of Newton–Cotes formulas.

The integrated indexes for the same two options of the projections have been computed with allowance made for the values of the areas calculated by formulae (29). Their values are given in Table 2.

Table 2.

Integrated indexes		
Integrated indexes	Option 1	Option 2
E1	0.7677	0.4134
E2	12.973	8.4495

The data in Table 2 also confirm the advantage of using the cylinder with an elliptical base.

The grids of the two options of the projections with the lines of equal distortion of both square measure scales and angular distortions are presented in Fig. 4 and Fig. 5.

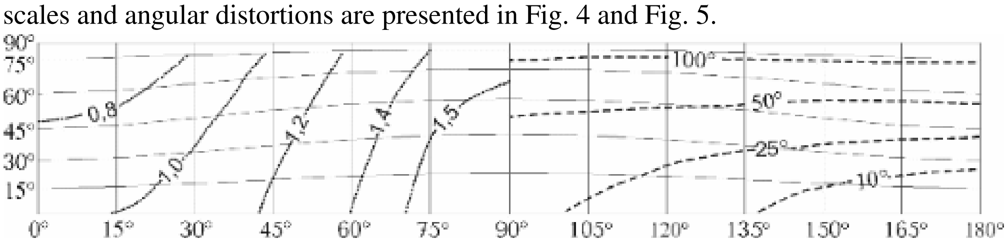

Fig. 4. A perspective cylindrical projection of a triaxial ellipsoid on the tangent circular cylinder.

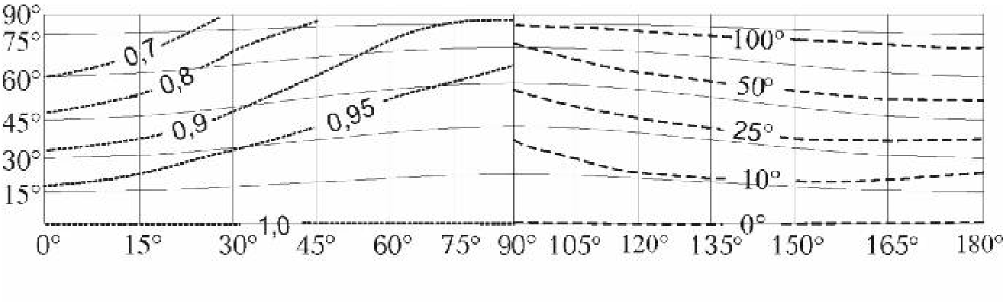

Fig. 5. A perspective cylindrical projection of a triaxial ellipsoid on the tangent cylinder with the elliptical base.

The application of a secant cylinder rather than a tangent one makes it possible to decrease distortions even more.

The approach suggested to the selection of the form of an auxiliary surface can be applied both to conic projections and to multiaxial ellipsoids.

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