HEALPix Mapping Technique and Cartographical Application

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Abstract

Hierarchical Equal Area isoLatitude Pixelization (HEALPix) system is defined as a layer which is used in modern astronomy for using pixelised data, which are required by developed detectors, for a fast true and proper scientific comment of calculation algoritms on spherical maps. HEALPix system is developed by experiments which are generally used in astronomy and called as Cosmic Microwave Background (CMB). HEALPix system reminds fragment of a sphere into 12 equal equilateral diamonds.

In this study aims to make a litetature study for mapping of the world by mathematical equations of HEALPix celling/tesellation system in cartographical map making method and calculating and examining deformation values by Gauss Fundamental Equations. So, although HEALPix system is known as equal area until now, it is shown that this celling system do not provide equal area cartographically.

Keywords: Cartography, Equal Area, HEALPix, Mapping

1. INTRODUCTION

Map projection, which is classically defined as portraying of world on a plane, is effected by technological developments. Considering latest technological developments, we can offer definition of map projection as “3d and time varying space, it is the transforming techniques of space’s, earth’s and all or part of other objects photos in a described coordinate system by keeping positions proper and specific features on a plane by using mathematical relation.”

Considering today’s technological developments, cartographical projections are developing on multi surfaces (polihedron) and Myriahedral projections which is used by Wijk (2008) for the first time. Within this scope, HEALPix which is developed for fast, true and scientific commentary on global sky mapping and uses unreal calgoritm is proposed for cartographic studies.

In this study Gorski et al. (1998b) and (2005) and HEALPix which is defined as usage of multipying the globe to 12 squares, will show that Calabretta’s design (2004) and Calabretta & Roukema (2007), which is said to area preserving and based on mathematical cell pixelisation, is not area preserving.

2. HEALPix

HEALPix is a software which has unreal algoritms and visual presentations, which is required via developed detectors in modern astronomy and used for scientific commentary of Cosmic

Microwave Background (CMB) data on global maps fastly and truely. HEALPix can be defined as a sky map format and visualisation analysis mass between information technologies and space units and has a wide range of use.

The best results are get when it provides below features for global sky mapping:

a- Hierarchical database usage
b- Equal Area
c- Isolatitude feature for harmonical assessments.

A single HEALPix provides above features and its naming is based on these three specifications.

Mathematical requirements which are needed by CMB are the basics of HEALPix (Gorski et al., 2005). Mathematical unit sphere [0, 1]*[0, 1] on in the basic of HEALPix is based upon mapping of each pixel in K*H. Here, K shows the pixel number between north and south poles, H shows pixels aroung the pole. As stated by Gorski et al., (1998b), O'Mullane et al., (2000) and Gorski et al., (2005) pixel borders are no geodesic and as pixel corners are not big circle arcs, pixels are not squares.

Considering sky map data standard, Gorski et al., (2005) has chosen H = 4, K = 3 as standard in his study in 2005. Total pixel number in standard application is 12. In other words, square is divided into 12 equal pixels. Standard choice;

a. Having less than 4 pixels in poles for avaiding narrow angles.
b. Fast harmonical applications in equatorial areas.
c. Minimising pixel shape change in equatorial areas to provide equal area feature.

HEALPix is understood as 12 devided pixels to give desired resolution and pixels 0-3 north, 4-7 equatorial and 8-11 south pixel numbered. Standard HEALPix application of globe to plane is seen on Figure 1. Calabretta (2004) and Calabretta & Roukema (2007) created this projection's cartographical application of such a study.

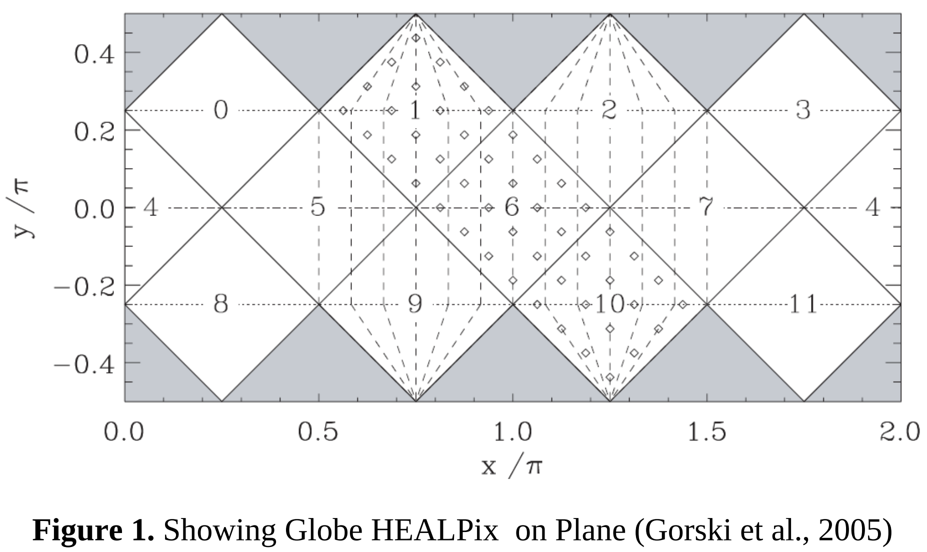

Figure 1. Showing Globe HEALPix on Plane (Gorski et al., 2005)

2.1.Cartographical Application

Cartographical application of HEALPix, is characteristically defined as below by Calabretta (2004) and Calabretta & Roukema (2007).

a. It is equal area; area on the globe is transferred to projection plane wtihout changing.
b. Parallels are horizontal plane lines as $\partial y/\partial \Phi = 0$
c. Parallels are always divided in the same way.
d. Pixels are in the area preserving equilateral quadrangles and considerd as $\pm 180^0$ in latitude-longitude network. For latitude longitute (Φ, λ) and for the cartesian coordinates on projection plane (x, y) are used.

In creating projection equalities, first Roukema & Lew (2004), Calabretta (2004) and Calabretta & Roukema (2007) stated the total areas 1/6 north and 1/6 as south pole, and the left 4/6 as equatoral area. This situation is HEALPix in Figure 2, and (K, H) = (3, 4) standard application. Φ_* is transition latitude between equatorial and polar regions. Polar and equatorial areas are restricted according to Φ_* transition latitude.

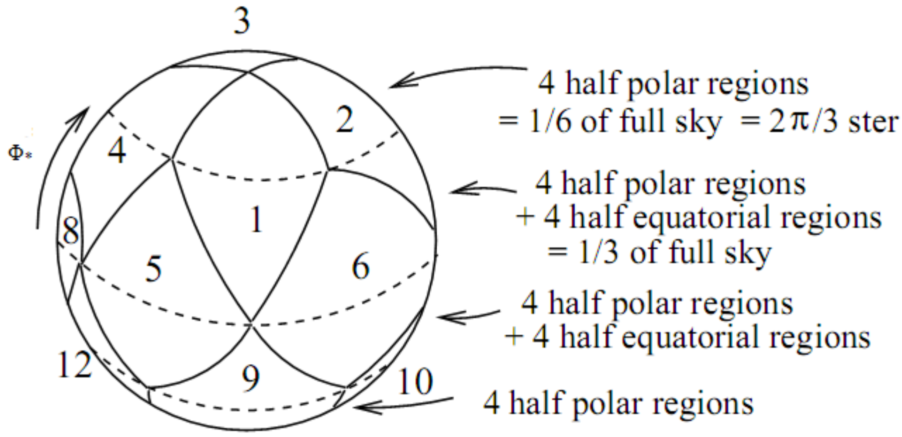

Figure 2. HEALPix (K, H) = (3, 4) applicaiton's standards illustration (Roukema & Lew, 2004).

Considering above stated definitions, HEALPix is the combination of Collignon projection's (1865) in pole areas, and Equal Area Cylindirc Projection of Lambert (1772) in equatoral parts (Collignon, 1865; Tissot, 1881; Lutque & Matarazzo, 2004; Calabretta & Roukema, 2007). HEALPix projection is not seen as documentation in cartography documents and not positioned in a web search. Especially, it does not take place in Snyder's (1993) revision of cartography history. MicroCAM web site of Illionis State University, commission cartography projections of international cartography union presents 320 projections in a catalogue (Anderson, 2003; Calabretta, 2004; Calabretta & Roukema, 2007). None of these projections have any resemblance with HEALPix projection. Equations which are created from these projections are used and Jacobian determinant and Gauss Fundamental Equations are used.

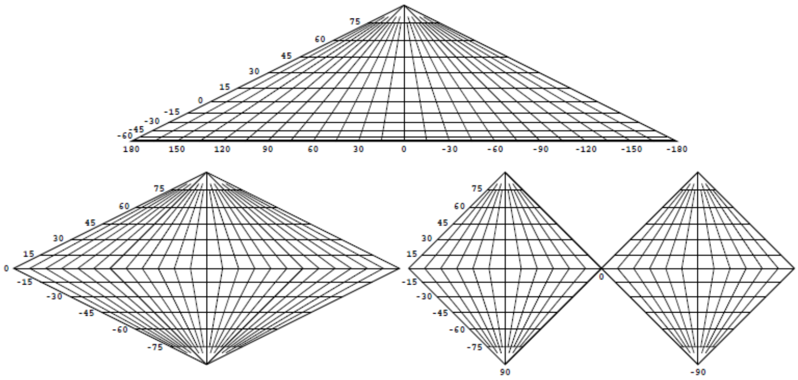

Figure3. EdouardCollignonprojections (Calabretta&Roukema, 2007).

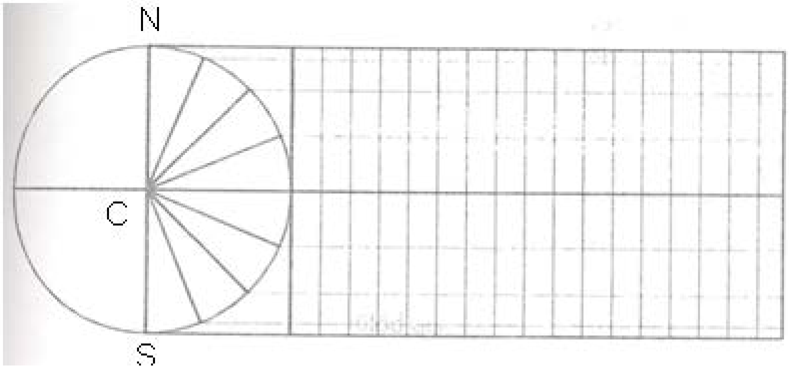

Figure 4. Lambert's Equal Area Cylindiric Projection (Yerci, 1997)

Gorski et al. (2005), has chosen $K = 3,\ H = 4$ application as standard application of numeric sky mapping in 2005. Calabretta (2004) and Calebretta & Roukema (2007) use K = 3 application in their studies.

2.2. Transition Latitude Φ_* Calculation

On sphere, steradian angle value which sees Φ_* should be 2π/3 which is 1/6 of the 4π steradian angle as seen in Figure 2. According to this information calculation is required;

$$\int_0^{2\pi} d\emptyset \int_{\Phi_*}^{\pi/2} \cos\Phi d\Phi = 2\pi/3$$

$$\Rightarrow 2\pi \sin[\Phi]_{\Phi_*}^{\pi/2} = 2\pi/3$$

$$\Rightarrow 2\pi(1 - \sin\Phi_*) = 2\pi/3$$

$$\Rightarrow sin\Phi_* = {}^{2}\!/_{3} \tag{1}$$

$$\Rightarrow \sin^{-1} 2/3 \approx 41.8^0 \tag{2}$$

2.3. Coordinate Calculations

HEALPix structure use octahedron as base (O'Mullane, 2000). Calabretta (2004) and Calebretta & Roukema (2007) configurated reference point as local coordinate systems origin and;

$$(\Phi_0, \lambda_0)_{HEALPix} = (0, 0) \tag{3}$$

In process, pixel's North, Equatoral and South parts should be checked and reletad parts formulas should be used.

2.3.1. Equatorial Region

In equatorial region $|\Phi| \leq \Phi_*$ and Lambert Equal Area Cylindirc Projection is used. Calabretta (2004) and Calebretta & Roukema (2007) gives relations to be used in equatorial region as below.

$$x = \frac{K\pi}{2H} \sin \Phi \tag{4}$$

$$y = \lambda \tag{5}$$

2.3.2. Polar Regions

In pole regions Φ $(> \Phi_*)$and Collignon projection is used. Calabretta (2004) and Calabretta & Roukema (2007) gives relations to be used in pole region as below.

$$x = \frac{\pi}{H}\left(\frac{K+1}{2} - \sigma\right) \tag{6}$$

$$y = \lambda_c + (\lambda - \lambda_c)\sigma \tag{7}$$

Here,

$$\sigma = \sqrt{K(1 - |\sin \Phi|)} \tag{8}$$

σ is the ratio of the distance of the pole from the paralel of Φ to that of the pole from the paralel transition latitude Φ_*.

$$\lambda_c = \frac{180^0}{H}\left(2\left\lfloor\frac{(\lambda + 180^0)H}{360^0} + \frac{1-\omega}{2}\right\rfloor + \omega\right) - 180^0 \tag{9}$$

$$\omega = \begin{cases} 1 \ldots K \; uneven \; number \; if \quad \Phi > 0 \\ 0 \ldots \qquad otherwise \end{cases} \tag{10}$$

Numerical results are get from (9) numbered relation. These formulas in South hemisphere are applied in equator by mirror symetry.

3. APPLICATION

We will check HEALPix's equal area feature by Gauss Fundamental Equation. At the same time we will map HEALPix standard application which is produced by us in VoteMap software, we will study on differences between spatial differences of pixels.

3.1. Control of Equal Area Feature by Gauss Fundamental Equation

The earth is either considered ellipsoid or sphere, it is impossible to transfer details on it to a map plane without deterioration and this deterioration is called as deformation. Deformation issues are mentioned by French Tissot (1881) for the first time.

- h: coefficient of deformation in the direction of meridian cycle (North-South direction) $(h = r_{\Phi})$
- k: coefficient of deformation in the direction of parallel $(k = r_{\lambda})$

shows and there is two perpendicular direction which length deformations reach extreme values (Ucar et al., 2004).

- a: Maximum length deformation coefficient $(a = r_{max})$
- b: Minimum length deformation coefficient $(b = r_{min})$

The directions in which length deformations are maximum and minimum are called as "Main Deformation Directions". On original surface, parameter curves, intersect each other perpendicularly. In case they intersect perpendicularly on the surface of projection, main deformation directions and parameter curve deformations are equal to each other. So;

$$\left.\begin{aligned} k &= a \\ h &= b \end{aligned}\right\} \tag{11}$$

Unit disc on the sphere transforms into ellipse shape while being transferred to projection plane. So transforming of a specific area to projection plane without being changed is called as equal area feature. Equal area condition which takes place in Gauss Fundamental Equatitons in Maling (1992) is used as "First Fundamental Quantities";

$$E = \left(\frac{\partial x}{\partial \Phi}\right)^2 + \left(\frac{\partial y}{\partial \Phi}\right)^2 \rightarrow h = \sqrt{E} \tag{12}$$

$$G = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 \rightarrow k = \sqrt{G}/\cos\varphi \tag{13}$$

In order to have equal area, it should be;

$$h \times k = a \times b = 1 \tag{14}$$

3.1.1 . Equatorial Region

If (4) and (5) numbered equations are put in (12) and (13) numbered formulas;

$$\frac{\partial x}{\partial \Phi} = \frac{K\pi}{2H}\cos\Phi \tag{15}$$

$$\frac{\partial x}{\partial \lambda} = 0 \tag{16}$$

$$\frac{\partial y}{\partial \Phi} = 0 \tag{17}$$

$$\frac{\partial y}{\partial \lambda} = 1 \tag{18}$$

If values required in (12) numbered equation are put in (15) and (17) equations;

$$E = \left(\frac{K\pi}{2H}\cos\Phi\right)^2 \rightarrow h = \frac{K\pi}{2H}\cos\Phi \tag{19}$$

is required. If values which are required from (13) relation are put in (16) and (18) relations;

$$G = 1 \rightarrow k = 1/\cos\Phi \tag{20}$$

If values which are required from (14) relation are put in (19) and (20) relations;

$$h \times k = 1$$

$$\Rightarrow \frac{K\pi}{2H}\cos\Phi \times \frac{1}{\cos\Phi} = 1$$

$$\Rightarrow \frac{K\pi}{2H} = 1$$

$$\Rightarrow H = \frac{K\pi}{2} \tag{21}$$

H value is required from (21) numbered relation which provides equal area feature. When K = 3 is chosen in HEALPix standard application,

$$H = \frac{3\pi}{2} = 4{,}71238898 \tag{22}$$

is found. When $H = 4{,}71238898$ and $H = 4$ is chosen, calculated coordinates are shown in Table 1. When $H = 4{,}71238898$ is chosen,it is seen in Table 2 that there is no area deformation. However, in application it is processed as H = 4 and naturally, area deformation occurs.

3.1.2Polar Regions

If values which are required from (12) and (13) relation are put in (6) and (7) relations;

$$\frac{\partial x}{\partial \Phi} = \frac{K}{2} \times \frac{\frac{\pi}{H}}{\sqrt{K(1-\sin\Phi)}} \times \cos\Phi = \frac{K}{2} \times \frac{\frac{\pi}{H}}{\sigma} \times \cos\Phi = \frac{K\pi}{2H\sigma}\cos\Phi \tag{23}$$

$$\frac{\partial x}{\partial \lambda} = 0 \tag{24}$$

$$\frac{\partial y}{\partial \Phi} = -\frac{K}{2} \times \frac{(\lambda - \lambda_c)}{\sigma} \times \cos\Phi \tag{25}$$

If $(\lambda - \lambda_c) = 0$ is accepted to fulfill condition of $\frac{\partial y}{\partial \Phi} = 0$, (25) numbered relation would be as below;

$$\frac{\partial y}{\partial \Phi} = -\frac{K}{2} \times \frac{(\lambda - \lambda_c)}{\sigma} \times \cos \Phi = 0 \tag{26}$$

$$\frac{\partial y}{\partial \lambda} = \sigma \tag{27}$$

If values which are required from (23) and (26) relation are put in (12) relation;

$$E = \left(\frac{K}{2} \times \frac{\frac{\pi}{H}}{\sigma} \times \cos \Phi\right)^2 \rightarrow h = \frac{K\pi}{2} \times \frac{\frac{\pi}{H}}{\sigma} \times \cos \Phi \tag{28}$$

is required. If values which are required from (24) and (27) relation are put in (13) relation;

$$G = \sigma^2 \rightarrow k = \frac{\sigma}{\cos \Phi} \tag{29}$$

When reqiured values from (26) and (27) relations are put in (12) numbered equation by condition of area preserving;

$$h \times k = 1$$

$$\Rightarrow \frac{K\pi \cos \Phi}{2H\sigma} \times \frac{\sigma}{\cos \Phi} = 1$$

$$\Rightarrow \frac{K\pi}{2H} = 1$$

$$\Rightarrow H = \frac{K\pi}{2} \tag{30}$$

H value which provides equal area is required from (30) numbered relation. When K = 3 is chosen in HEALPix standard application;

$$H = \frac{3\pi}{2} = 4{,}71238898 \tag{31}$$

When $H = 4{,}71238898$ ve $H = 4$ is chosen, calculated coordinates are shown in Table 1. When $H = 4{,}71238898$ is chosen, it is shown in Table 2 that there is no area deformation. However, in the application it is applied as H = 4 and naturally deformation occurs.

Table 1. Cartesian coordinates (x, y) for geographical coordinates for HEALPix standard application $(K, H) = (3, 4)$ and $(K, H) = (3, 4.71238898)$

Geographical Coordinates	H = 4 K = 3	H = 4,71238898 K = 3
$\boldsymbol{\varphi = 0}$ $\boldsymbol{\lambda = 0}$	$x = 0$ $y = 0$	$x = 0$ $y = 0$
$\boldsymbol{\varphi = 0}$ $\boldsymbol{\lambda = 15^0}$	$x = 0$ $y = 0{,}261799387$	$x = 0$ $y = 0{,}261799387$
$\boldsymbol{\varphi = -15^0}$ $\boldsymbol{\lambda = -15^0}$	$x = -0{,}304914004$ $y = -0{,}261799387$	$x = -0{,}258819045$ $y = -0{,}261799387$
$\boldsymbol{\varphi = 15^0}$ $\boldsymbol{\lambda = -15^0}$	$x = 0{,}304914004$ $y = -0{,}261799387$	$x = 0{,}258819045$ $y = -0{,}261799387$
$\boldsymbol{\varphi = 41^0, 8103}$ $\boldsymbol{\lambda = 0}$	$x = 0{,}785397935$ $y = 0$	$x = 0{,}666666472$ $y = 0$
$\boldsymbol{\varphi = 50^0}$ $\boldsymbol{\lambda = 15^0}$	$x = 0{,}912809511$ $y = 0{,}346740286$	$x = 0{,}774816778$ $y = 0{,}250434102$
$\boldsymbol{\varphi = -50^0}$ $\boldsymbol{\lambda = 15^0}$	$x = -0{,}912809511$ $y = 0{,}346740286$	$x = -0{,}774816778$ $y = 0{,}250434102$
$\boldsymbol{\varphi = -50^0}$ $\boldsymbol{\lambda = -15^0}$	$x = -0{,}912809511$ $y = -0{,}346740286$	$x = -0{,}774816778$ $y = -0{,}250434102$
$\boldsymbol{\varphi = 90^0}$ $\boldsymbol{\lambda = 0}$	$x = 1{,}570796327$ $y = 0{,}127411348$	$x = 1{,}333333333$ $y = 0{,}031105163$

3.2 VoteMap

VoteMap programme is created by C++ and not only for one projection but for all projections which can be expressed by mathematical equatitons. In programme, data about the chosen projection, and related mathematical equations, parameters, statics and drawing is done by standard values which are valid for all projections.

When VoteMap programme is used in HEALPix $(K, H) = (3, 4)$, cartographically view of Figure 5 is required. By the help of VoteMap programme, as in Calabretta (2004) and Calabretta & Roukema (2007) 12 pixels studies, Figure 6 view is obtained. It is seen that in Figure 7 as pixels are not exactly equilateral quadrangles, transition latitude, creates a trapezoid with 45 degree latitude and this value is called as “Overlap Angle” by us. Using Figure 6 and Figure 7, we can see the deformation differences in regards of km^2 in Table 3.

Table 2. Cartesian coordinates (x, y) for geographical coordinates for HEALPix application$(K, H) = (3, 4.71238898)$ and Area Calculation on Sphere

Geography Coordinates (φ, λ)	Cartesian Coordinates (x, y)	Area Calculation $2S = \sum (X_n Y_{n+1} - Y_n X_{n+1})$	Area Calculation on Sphere $F = 2R^2 \frac{\lambda}{\rho} \sin \frac{\varphi_1 - \varphi_2}{2} \cos \frac{\varphi_2 + \varphi_1}{2}$
$1(30^0, 15^0)$	$x_1 = 0{,}5000000$ $y_1 = 0{,}261799387$		
$2(30^0, 30^0)$	$x_2 = 0{,}500000$ $y_2 = 0{,}523598775$	**0,063141026**	**0,063141026**
$3(15^0, 30^0)$	$x_3 = 0{,}258819045$ $y_3 = 0{,}523598775$		
$4(15^0, 15^0)$	$x_4 = 0{,}258819045$ $y_4 = 0{,}261799387$		
$1(40^0, 15^0)$	$x_1 = 0{,}642787609$ $y_1 = 0{,}261799387$		
$2(40^0, 30^0)$	$x_2 = 0{,}642787609$ $y_2 = 0{,}523598775$	**0,037381708**	**0,037381708**
$3(30^0, 30^0)$	$x_3 = 0{,}5000000$ $y_3 = 0{,}523598775$		
$4(30^0, 15^0)$	$x_4 = 0{,}5000000$ $y_4 = 0{,}261799387$		
$1(75^0, 15^0)$	$x_1 = 1{,}120184954$ $y_1 = 0{,}214140029$		
$2(75^0, 30^0)$	$x_2 = 1{,}120184954$ $y_2 = 0{,}297843202$	**0,026153869**	**0,026153869**
$3(60^0, 30^0)$	$x_3 = 0{,}910683602$ $y_3 = 0{,}4022130281$		
$4(60^0, 15^0)$	$x_4 = 0{,}910683602$ $y_4 = 0{,}23615612$		

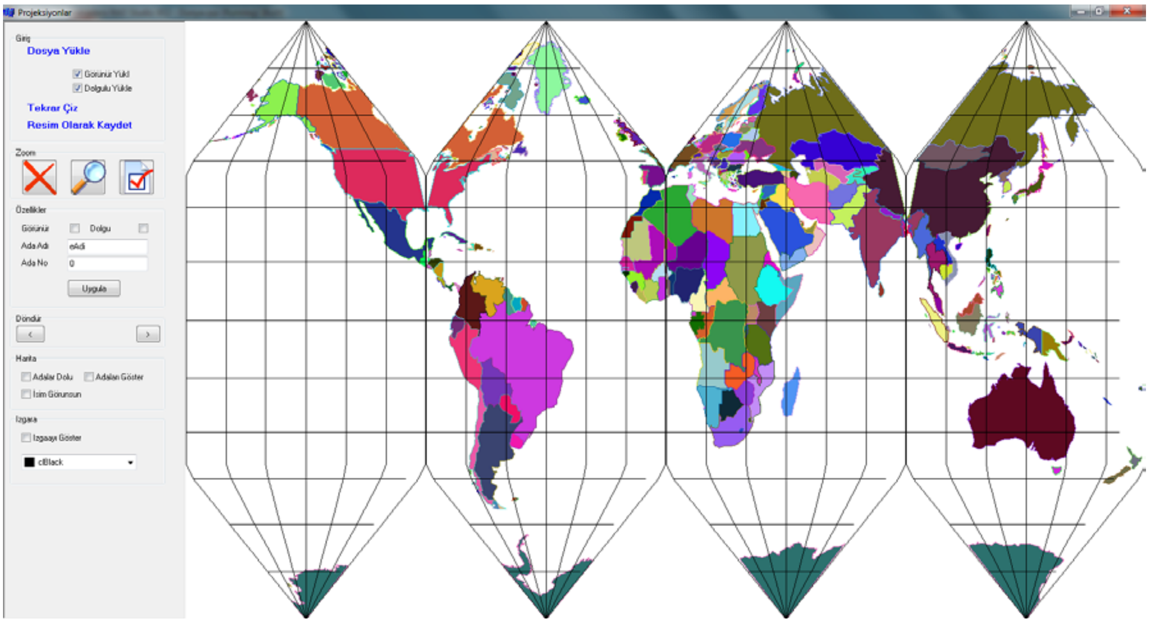

Figure 5. Required view of HEALPix (K, H) = (3, 4) application with VoteMap programme

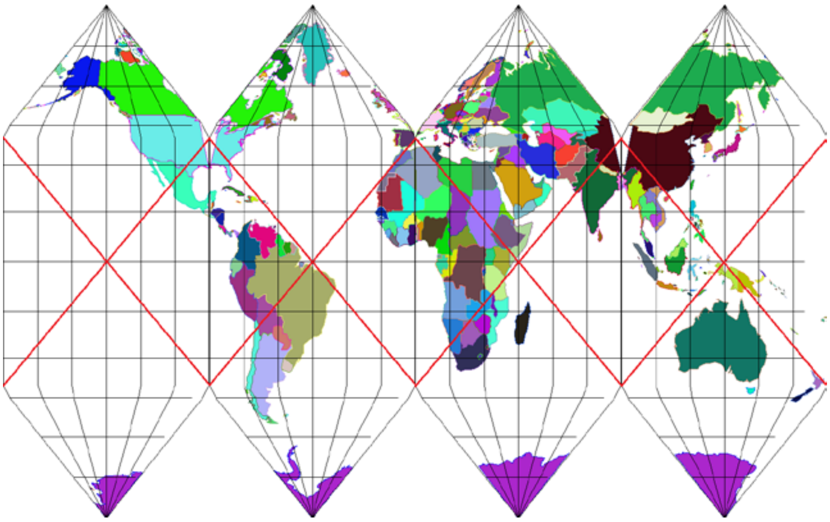

Figure 6. 12 pixels view of HEALPix and its required view from VoteMap

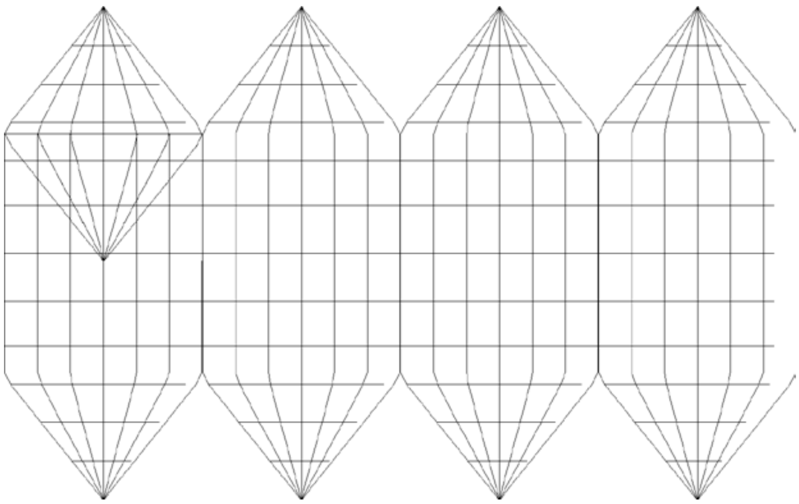

Figure 7. Showing of Overlap Angle in HEALPix $(K, H) = (3, 4)$ application

Table 3. HEALPix $(K, H) = (3, 4)$ Deformation, Polar and Equatorial Pixel Difference Values.

	H = 4, K = 3
a	16,054619
b	1,185253
a*b (Area Deformation)	18,849556
Polar Area	68445,5
Equatorial Area	68040
Areal Difference	405,5
Σ Polar Area	547564
Σ Equatorial Area	272160
Σ Area	819724
Σ Areal Difference	1622
Ratio (Σ Areal Difference / Σ Area)	0,001979
Overlap Angle	2,378049
Σ Areal Difference (km^2)	1009419,197

4. RESULTS

As a result of this study about HEALPix $(K, H) = (3, 4)$ which is mostly used variation in astronomical cartography studies and created by mathematical base of software is studied. As seen in studies, HEALPix does not provide equal area cartographically. At the same time, it does not consist of same shaped and equilateral quadrangle pixels.

As a result of our studies on HEALPix, we think that HEALPix $(K, H) = (3, 4)$ application does not provide desired features and should not be used. As we think that all studies has a contribution to scientific development, we believe that it will contribute to cartographical applications in future. As a result of our studies we believe that Hierarchical Equal Area IsoLatitudes Pixelisation (HEALPix) should be renamed as Hierarchical isoLatitude Pixelisation (HiLPix).

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